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by

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Abstract

The non-stationary kinetic theory of Israel and Stewart [16] is extended from the case of a gas consisting of only one species of particle to the case where the gas consists of an arbitrary number of species. A particular species may have non-zero or zero rest mass (e.g. electrons or photons). It will be shown that generalized fitting conditions, which determine the temperature and relativistic chemical potential of each species, are required to adequately treat the physics of the gas, which nevertheless will be independent of the choice of the fitting conditions to first order. Equations which describe the transport of thermal and viscous effects will be derived; and coefficients of thermal coductivity, bulk viscosity, and shear viscosity will be defined. We shall also obtain the entropy production for any particle species.

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I. Introduction

Irreversible relativistic thermodynamics has undergone, in the last fourteen years, an extension of its applicability. Originally, it was a stationary theory in which gradients of the macroscopic deviations from equilibrium, such as heat flux and viscous stresses, were considered negligible on the scale of the mean free path. However, this theory had one great defect. It predicted infinite speeds of propagation of thermal and viscous effects whereas we would expect such speeds to be about the mean molecular speed and certainly not greater than the speed of light (Israel[14]).

Early attempts at resolving this problem were focused on the heat transport equation (Fourier's law) and consisted of the addition of ad hoc terms to convert it from a parabolic to a hyperbolic differential equation (infinite versus finite propagation speeds respectively). The suggestions of Cattaneo [5] and Vernotte [39] were further discussed by Kranys [20]. These authors proposed that a term proportional to the time derivative of the heat flux be added to Fourier's law and showed that the resulting equation was hyperbolic with finite propagation speeds.

In a paper on non-relativistic thermodymamics, Muller [28] suggested that the expression for the entropy was incomplete and, in addition to the number density, pressure, and energy density, must include as independent variables the heat flux and viscous stresses. This suggestion is in fact the foundation of the successful relativistic theory and is essentially an extension of the theory from the stationary to the non-stationary regime. Muller went on to show that hyperbolic equations of transport resulted as a consequence of this formalism.

Subsequent discussions of the propagation speed problem focussed on the non-stationary aspect of the relativistic theory via the relativistic Grad method of moments. Kranys [21,22] extended Chernikov's [7,8] thirteen moment stationary theory to the non-stationary case and obtained hyperbolic propagation speeds. Stewart [33] showed that by retention of non-stationary terms in the fourteen-moment theory, the upper bound on propagation speeds was $\sqrt{3/5}$ c. Discussions of the fourteen moment case were also presented independently by Marle [27] and Kranys [23].

The discussions cited above were in the realm of kinetic theory. Within the context of relativistic phenomenological theory, however, the problem of a causal theory which predicted finite propagation speeds was not resolved until Israel [14] independently rediscovered and extended the idea of Muller to the relativistic case. This theory was later combined and compared with the fourteen moment kinetic theory by Israel and Stewart [15,16]. An analysis of the propagation speeds of thermal and viscous effects by an examination of the characteristics of the transport equations was reported by Stewart [34] and Israel and Stewart [17]. An analysis via Fourier analysis of the transport equations was performed by Kranys [24]. The upper bound on the speed of propagation of thermal and viscous effects appears to be established as $\sqrt{3/5}\, c$ [16]. The extension of this theory to polarizable media in the presence of electric and magnetic fields has been performed. We shall not discuss that theory herein but refer the reader to a recent article by Israel and Stewart [18] which performs this extension and also discusses the background of this theory.

In summary, we have at the present time a phenomenological theory for relativistic simple media and mixtures which takes into account transient effects and predicts finite propagation speeds of thermal and viscous effects; we also have a relativistic transient kinetic theory for single species gases. There are astrophysical situations, however, where this latter theory is not applicable because we have to deal with plasmas. For example, such situations are: the accretion of matter through the boundary between a star and a hypothetical neutron star or black hole in the star's core [38]; accretion disks around neutron stars or black holes; and the leptonic era in the early history of the universe. In these situations we imperatively require an extension of transient kinetic theory to the case of mixtures. This extension is performed in this thesis. Furthermore; we apply the extended theory to a case of special astrophysical interest: a mixture of matter and zero-mass particles (photons or neutrinos); that is, we study relativistic radiative transport with the inclusion of transient effects.

To illustrate the need for transient theory as opposed to quasi-stationary theory in astrophysical situations, we shall briefly discuss accretion disks and the leptonic era in the early history of the universe. Consider a black hole which is accreting matter. If the accreting matter has a large amount of angular momentum then an accretion disk is formed [4]. The principal model discussed in the literature is the thin disk model where the half thickness of the disk, h, is much smaller than the distance to the black hole, r, i.e. $h/r \ll 1$. The matter in the disk moves in Keplerian orbits and viscosity between adjacent rings transports angular momentum outward and matter inward. The viscosity also acts as the principal source of heating in the disk. Most of the radiation is produced in a small region near the black hole and ionizes the infalling matter to distances beyond the accretion radius of the black hole. Thus the disk consists wholely of a highly ionized plasma. The disk has three zones of interest: the outer zone where the pressure is dominated by the matter pressure and the major source of opacity is due to free-free collisions (Bremsstrahlung); the middle region where gas pressure still dominates but the major source of opacity is due to electron scattering; and the inner zone where the radiation pressure dominates and opacity is due mainly to electron scattering.

The vertical structure of the disk is determined by a balance between compressional tidal forces and the outward pressure and radiation flux. The optical depth τ is given by $\tau = h/\lambda$ [25] where λ is the mean free path of the photons. The half thickness of the disk, h, is a distance scale of instabilities and fluctuations in the disk [25,32]. Normally a stationary radiative diffusion equation is employed to help find the vertical structure of the disk. In the optically thick case, $\tau \gg 1$ and in the optically thin case, $\tau \ll 1$ this equation is valid; however, when $\tau \approx 1$ that is, the disk is neither optically thick or thin, then we are in the non-stationary regime and we must use transient theory.

Novikov and Thorne [30] have developed a model of a thin disk. In this model they note that in the outer and middle zones, the disk is optically thick. However they specify a solution in which a small region in the inner zone is optically thin. In this model therefore, there must be a transition zone between the optically thick and optically thin regions where we must employ transient theory to determine the vertical structure. Eardly et al. [9] have also constructed a disk model in which they find the inner region of the disk to be marginally optically thin to electron scattering. Hence we should apply transient theory to this model as well. Other models may also require transient theory because thin disk models are secularly unstable in the inner zone so that the thin disk model may be invalid in the inner zone [26]. The disk probably becomes, in this region, a small cloud around the black hole; in this case the optical depth could be close to one. Also, convective turbulence may cause the disk to thicken so that the half thickness of the disk could become the same size as the photon mean free path [19]. We conclude therefore that transient theory is a valid or a necessary approach to accretion disk physics.

The history of the universe is usually divided into four eras: the hadronic era when the temperature, T, was greater than 10^{12} °K ; the leptonic era when $10^{12}\ °K > T > 10^{10}\ °K$; the radiation dominated era when $10^{10}\ °K > T > 10^{4}\ °K$; and the matter dominated era when $T < 10^{4}\ °K$. This last era comprises 99.9% of the history of the universe. We restrict our attention to the leptonic era. At this time the universe consisted of a mixture of electrons, muons, neutrinos, photons, and their anti-particles. We can estimate the temperature when transient effects become important by comparing the rate of expansion of the universe, H, to the rate of interactions , $\sigma_{wk}n$, where σ_{wk} is the weak interaction cross-section and n is the lepton density. As the universe expands the particles will tend to go out of equilibrium with each other but their interactions tend to restore that equilibrium. Equilibrium will be maintained when $\sigma_{wk}n/H > 1$ (rate of expansion is less than rate of interactions) but equilibrium will not be maintained when $\sigma_{wk}n/H < 1$. We conclude that transient theory will become important when $\sigma_{wk}n/H \simeq 1$.

Weinberg [43] has computed this ratio:

$$\frac{\sigma_{wk} n}{H} = \left\{ \frac{T}{10^{10}\ {}^\circ K} \right\} \exp\left\{ - \frac{10^{12}\ {}^\circ K}{T} \right\} \quad . \qquad (1.1)$$

This ratio is unity for $T \simeq 1.3 \times 10^{11}\ {}^\circ K$. Hence transient effects will become important when the temperature of the universe drops below this value.

We shall now summarize the phenomenological theory for a simple (single component) fluid in a gravitational field; this summary allows us to illustrate the principal features of the non-stationary theory and to identify the essential difference between quasi-stationary and non-stationary theory. An arbitrary state of the fluid is described by three primary variables. These are the entropy flux S^α, the energy-momentum tensor $T^{\alpha\beta}$, and the number flux N^α. The number flux and the energy-momentum tensor are assumed to be conserved and the entropy production is assumed to be positive:

$$N^\alpha{}_{|\alpha} = 0 \;\; ; \;\; T^{\alpha\lambda}{}_{|\lambda} = 0 \;\; ; \;\; S^\alpha{}_{|\alpha} \geq 0 \;\; ; \qquad (1.2)$$

where the single stroke denotes covariant differentiation.

The equilibrium state of the fluid (denoted by a superscript 0) is characterized by four properties. First, the entropy production is zero: $\overset{\circ}{S}{}^\alpha{}_{|\alpha} = 0$. Secondly, there exists a unique time-like unit four vector u^α which we call the flow vector, such that we have

$$\overset{\circ}{S}^{\alpha} = Su^{\alpha} \; ; \; \overset{\circ}{N}^{\alpha} = nu^{\alpha} \; ; \; \overset{\circ}{T}^{\alpha\beta} = \rho u^{\alpha} u^{\beta} + P\Delta^{\alpha\beta} \; ; \qquad (1.3)$$

that is, the entropy flux, number flux, and the energy-momentum tensor are spatially isotropic. Here S is the entropy, n is the number density, ρ is the energy density, P is the pressure, and $\Delta^{\alpha\beta} \equiv u^{\alpha}u^{\beta} + g^{\alpha\beta}$ is the spatial projection operator of u^{α}. Thirdly, each equilibrium state is characterized by an equation of state $S=S(n,\rho)$ which determines the entropy and from which we can find the pressure by

$$S = (\rho + P)/T - \alpha n \; , \qquad (1.4)$$

where T is the temperature and α is the relativistic chemical potential of the equilibrium state. Finally, the flow vector u^{α} is shear free and expansionless; and the relativistic chemical potential is constant:

$$u^{\alpha}{}_{|\alpha} = 0 \; ; \; \Delta_{\alpha}{}^{\lambda} \Delta_{\beta}{}^{\tau} u_{(\lambda|\tau)} = 0 \; ; \; \alpha_{|\mu} = 0 \; . \qquad (1.5)$$

The covariant formulation of the equilibrium form of the entropy flux may be expressed as

$$\overset{\circ}{S}^{\mu} = P\beta^{\mu} - \alpha\overset{\circ}{N}^{\mu} - \beta^{\lambda}\overset{\circ}{T}_{\lambda}{}^{\mu} \; , \qquad (1.6)$$

where $\beta^\lambda \equiv u^\lambda/T$. Equation (1.6) implies that

$$d\mathring{S}^\mu = -\alpha d\mathring{N}^\mu - \beta_\lambda d\mathring{T}^{\lambda\mu} , \tag{1.7}$$

where the differentials are constrained to displacements between equilibrium states.

To investigate non-equilibrium states we release the constraints on the differentials in equation (1.7) and assume that they apply to displacements from equilibrium states to arbitrary nearby states:

$$dS^\mu = -\alpha dN^\mu - \beta_\lambda dT^{\lambda\mu} . \tag{1.8}$$

Here α and β^λ are the variables of a nearby equilibrium state such that the deviations from equilibrium $N^\alpha - \mathring{N}^\alpha$ and $T^{\alpha\beta} - \mathring{T}^{\alpha\beta}$ are small quantities compared to N^α and $T^{\alpha\beta}$ and are said to be of first order. Equation (1.8) suggests that equation (1.6) be generalized to

$$S^\mu = P\beta u^\mu - \alpha N^\mu - \beta_\lambda T^{\lambda\mu} - Q^\mu , \tag{1.9}$$

where P is the pressure of a nearby equilibrium state and Q^μ is some unspecified second order term.

Now the essential difference between quasi-stationary theory and non-stationary theory resides in the treatment of Q^μ . In general, Q^μ will depend in some fashion on the macroscopic deviations from equilibrium. If these deviations from equilibrium vary over large distance scales compared to the mean free path, then their space-time gradients are negligible to first order and hence $Q^{\mu}_{|\mu}$ will be negligible to second order. Therefore $Q^{\mu}_{|\mu}$ will not contribute to the entropy production which is calculated to second order; and we may therefore neglect Q^{μ} in equation (1.9). In non-stationary theory, however, the deviations from equilibrium vary over a length scale comparable with the mean free path. Consequently their space-time gradients are not negligible to first order and then $Q^{\mu}_{|\mu}$ is not negligible to second order. In this case, $Q^{\mu}_{|\mu}$ contributes to the entropy production and neglect of Q^{μ} in equation (1.9) is not justified; that is, in the non-stationary theory, Q^{μ} must be retained.

In equilibrium, the unit time-like vector parallel to the number flux, u^{α}_{N}, and the unit time-like eigenvector of the energy-momentum tensor, u^{α}_{E}, coincide. In non-equilibrium they differ by a small angle of first order. We can choose an arbitrary velocity u^{α} within a small cone whose angle is of first order and includes u^{α}_{N} and u^{α}_{E} . Then the number density, the energy density, the pressure, and the entropy are independent of the choice of u^{α} to first order. Furthermore, for a given choice of u^{α} we can decompose the number flux and energy-momentum tensor uniquely:

$$\begin{aligned} &N^{\mu} = nu^{\mu} + j^{\mu} \ , \ j^{\mu}u_{\mu} = 0 \ ; \\ &T^{\mu\nu} = \rho u^{\mu}u^{\nu} + (P + \pi)\Delta^{\mu\nu} + h^{\mu}u^{\nu} + h^{\nu}u^{\mu} + \pi^{\mu\nu} \ , \qquad (1.10) \\ &u_{\lambda}h^{\lambda} = 0 \ , \ u_{\lambda}\pi^{\lambda\mu} = 0 \ , \ \pi^{\lambda}{}_{\lambda} = 0 \ ; \ \Delta^{\mu\nu} = u^{\mu}u^{\nu} + g^{\mu\nu} \ . \end{aligned}$$

Here j^{α} is particle drift, h^{α} is the momentum flux, π is the bulk stress, $\pi^{\alpha\beta}$ are the viscous stresses, and $\Delta^{\alpha\beta}$ is the spatial projection operator of u^{α}. The particle drift and momentum flux change in first order if the choice of u^{α} changes. Consider however, the heat flux q^{α} which we define as the energy flux relative to the particle flow; it is given by

$$q^{\alpha} = h^{\alpha} - \frac{\rho + P}{n} j^{\alpha} \ . \qquad (1.11)$$

Then if the choice of u^{α} changes, q^{α} is unchanged to first order.

A crucial aspect of transient phenomenological theory is the specification of Q^{μ} in equation (1.9). To obtain linear phenomenological laws it is sufficient to assume that Q^{μ} is a quadratic function of the macroscopic deviations from equilibrium π, q^{α} etc. In particular, the simplest approximation which leads to hyperbolic phenomenological laws is the hydrodynamical description which assumes that an arbitrary state of the gas close to equilibrium can be specified completely by the number flux and the energy-momentum tensor. Hence, the entropy flux is a function solely of the number flux and the energy-momentum tensor. In this case Q^{α} is given by

$$\begin{aligned} TQ^{\mu} = \frac{1}{2} u^{\mu} (\beta_0 \pi^2 + \beta_1 q_{\lambda} q^{\lambda} + \beta_2 \pi^{\alpha\beta} \pi_{\alpha\beta}) \\ - \alpha_0 \pi q^{\mu} - \alpha_1 q^{\lambda} \pi_{\lambda}{}^{\mu} + \frac{T}{\rho + P} (\frac{1}{2} h^{\lambda} h_{\lambda} u^{\mu} + h^{\lambda} \pi_{\lambda}{}^{\mu}) , \end{aligned} \qquad (1.12)$$

where α_0, α_1, β_0, β_1, and β_2 are undetermined thermodynamical functions. We require that $u_{\mu} Q^{\mu} \leq 0$ which implies, for all states characterized by a given number and energy density, that equilibrium has the largest entropy; furthermore, it can be show to imply that the fluid is dissipative, that is, it has positive relaxation times.

The phenomenological laws (transport equations) may be inferred from the requirement that the entropy production is postive, $S^{\alpha}{}_{|\alpha} \geq 0$ if we assume there is a linear relationship between π, q_{λ}, and $\pi^{\alpha\beta}$ and their gradients, the gradients of the thermodynamical variables and the flow vector, the shear, and the volume expansion. Consequently with the choice $u^{\alpha} = u^{\alpha}_{E}$ we obtain

$$\begin{aligned} \pi &= - \frac{1}{3} \zeta_V (u^{\alpha}_{E|\alpha} + \beta_0 \dot{\pi} - \alpha_0 q^{\alpha}{}_{|\alpha}) ; \\ q^{\mu} &= \kappa T \Delta^{\mu\lambda} (\frac{nT}{\rho + P} \alpha_{|\lambda} - \beta_1 \dot{q}_{\lambda} + \alpha_0 \pi_{|\lambda} + \alpha_1 \pi_{\lambda}{}^{\gamma}{}_{|\gamma}) ; \qquad (1.13) \\ \pi_{\mu\nu} &= -2 \zeta_S (u_{E<\mu|\nu>} + \beta_2 \dot{\pi}_{\mu\nu} - \alpha_1 q_{<\mu|\nu>}) ; \end{aligned}$$

where ζ_V is the bulk viscosity, κ is the thermal conductivity, ζ_S is shear viscosity, and the angular brackets denote the trace free spatial part of the tensor enclosed by them. If we had chosen $u^\alpha = u^\alpha_N$ we would have obtained a set of equations similar to those above but with some of the coefficients slightly changed. These equations are hyperbolic and predict finite propagation speeds.

The kinetic theory of a single species gas duplicates the features presented above for the phenomenological theory. Briefly, kinetic theory postulates a distribution function for the gas and the Boltzmann equation which governs the distribution function. Then, in terms of this distribution function, the number flux, the energy-momentum tensor, and the entropy flux can be defined. The requirement $S^\alpha{}_{|\alpha} = 0$ for equilibrium uniquely specifies the distribution function as a function of the temperature, the relativistic chemical potential, and the flow vector u^α . The transition to non-equilibrium states of the gas close to equilibrium is achieved via the relativistic Grad fourteen moment approximation method, which assumes that the distribution function is expressable in terms of an equilibrium distribution function and a first order quadratic function of the particle four-momentum. We can then find the distribution function in terms of the structures given in equations (1.10) and (1.11). Consequently the entropy flux is found to have the structure given by equations (1.9) and (1.12). The transport laws may be derived and come from the conservation laws for number flux and energy-momentum and the balance equation for the third tensor moment of the distribution function, which we herein call the double-momentum flux. An important aspect of the kinetic theory as opposed to the phenomenological theory is that we can actually specify the functional form of the transport coefficients and relaxation times α_0, α_1, β_0, β_1, and β_2 for a gas.

The problem considered in this thesis is the extension of the relativistic kinetic theory, as presented in the formulation of Israel and Stewart [16], from the single species gas to a gas which contains many particle species. We do not wish to restrict the analysis to species which have only non-zero rest mass, but also wish to consider species which have zero rest mass, such as photons and neutrinos. The theory of Israel and Stewart is suited only for non-zero rest mass particle species; however, we shall extend the theory in an appropriate fashion to handle zero mass particle species.

One might expect that we shall find the structure of the entropy flux to be similar to equations (1.9) and (1.12) for each component of the gas. Futhermore, one might expect that we should obtain transport laws that resemble equation (1.13) although of a more complex nature. Also, one might expect that there are more functions to be specified like α_0, α_1, etc. Indeed, the derivation of all of these details is a major task to be carried out in this thesis.

We shall briefly outline the format of the thesis and the major points of each chapter. The main body of our thesis consists of six chapters plus two appendices. Chapter II consists of the fundamentals of Boltzmann kinetic theory. This consists of specifying the Boltzmann equation and deriving the master balance equation, special cases of which are the mass and number fluxes, the energy-momentum tensor and the entropy flux. Chapter III applies this theory to the equilibrium situation. The two important aspects of this chapter are the introduction of the thermodynamic functions necessary for all of the subsequent analysis of the physics, and the equilibrium structures of the number flux, the energy-momentum tensor, and the entropy flux. The material presented in these two chapters is not new but its presentation is necessary to provide the foundation upon which the subsequent analysis relies.

In chapter IV we analyse the non-equilibrium situation of the gas via the Grad fourteen moment approximation. It is here that we shall introduce the idea of fitting and frame changes under which the mathematical description of the physics will be shown to be invariant to first order. Also the deviations from equilibrium are solved for, in terms of physical quantities such as heat flux and the viscous stresses, by a method which differs from the approach normally used (Israel[12], Israel and Stewart [15,16]). This approach leads to a more aesthetic and compact notation. In terms of this solution, we shall then obtain an expression for the entropy flux.

Chapter V examines the non-stationary (transient) aspects of the theory. It is here that we compute the derivatives of physical quantities such as the mass flux and the energy-momentum tensor. We also compute the derivative of the entropy flux for any particle species, that is, we compute the entropy production for each species in the gas. These computations proceed by a different, although equivalent, approach than that of Israel and Stewart [16] for the single species case. Furthermore, the equations of transport of thermal and viscous effects for non-zero rest mass particle species are derived; and the coefficients of thermal conductivity, bulk viscosity, and shear viscosity for each species are defined.

Chapter VI considers the problem of extending the Grad method of moments to zero rest mass particle species such as photons and neutrinos. The solution for the deviations from equilibrium in terms of physically meaningful quantities leads to, when inserted into the Boltzmann equation, the non-stationary transport equations. Finally, we examine the entropy flux and define the coefficients of heat conductivity, shear viscosity, and bulk viscosity for zero rest mass particle species. The scenario envisioned in this chapter is a mixture of matter and radiation, that is, this chapter deals with transient radiative transfer. We noted earlier that this situation is of special astrophysical interest and would be applicable to important situations, e.g. accretion disks and the lepton era of the universe.

Chapter VII deals with the computation of quantities which depend upon the collision cross sections and the deviations from equilibrium; these quantities are necessary for the solution of the transport equations. This chapter should really be regarded as a super appendix because it deals with a very complicated analysis to provide minor although necessary information for the discussion of the physics.

The first appendix deals with the functional form of the thermodymanic functions introduced in chapter III. Various relationships between these functions are cited and some other frequently used results are stated.

The second appendix is a list of the symbols employed in this thesis. It is arranged alphabetically, Latin letters first followed by Greek letters. The list is necessary because so many symbols are used that the reader may lose track of the meaning of a particular symbol. Instead of trying to find the meaning in the text, all he or she has to do is refer to this list for a quick reminder.

As a final note, we declare our space-time convention. In a local Lorentz frame the metric takes the following form:

$$g_{\alpha\beta} \simeq \eta_{\alpha\beta} \equiv \text{diag}\ (1,1,1,-1) \quad . \tag{1.14}$$

The space-time coordinates x^{α} have the Euclidean-Minkowski form

$$x^\alpha = (x,y,z,ct) \quad ; \quad \alpha = 1, 2, 3, 4. \qquad (1.15)$$

As a consequence of this convention, four vectors are time-like, null, or space-like if their lengths are negative, zero, or positive, respectively. Finally, throughout this thesis, we shall adopt units such that the speed of light, c, and Planck's constant, h, are unity (h=c=1).

II. Boltzmann Kinetic Theory

All of the calculations in this thesis are performed within the context of Boltzmann kinetic theory. Before we can examine the main topics of this thesis we must present the notation employed and the fundamentals of the kinetic theory; this is the purpose of this chapter.

First we shall present the notation employed and some basic definitions. Then we shall define the distribution function N_A and present the Boltzmann equation as the equation which governs the evolution of the distribution function; at this point we shall also discuss the role of collisions of the gas particles. Once we have specified the Boltzmann equation we may derive from it a master balance equation which is then applied to find the balance equations for suitably defined quantities such as energy-momentum and entropy. Finally we discuss how these quantities may be decomposed with respect to an arbitrary comoving observer.

A. The Boltzmann Equation

Let us consider a multi-component gas. Each component of this gas is a particle species which we will identify by an Arabic number. These numbers will be generically represented by capital letters, for example, A = 1,2,3 etc., and used as subscripts on our mathematical symbols.

A particle which is a member of species A will have a mass $\overset{\circ}{m}_A$, a charge e_A, and a spin s_A (plus any other distinguishing characteristics) which are common to all particles of species A. Each particle has its own world-line $x^\alpha_A(\tau_A)$ where τ_A is the world line parameter; we assume that τ_A is an affine parameter for geodesics. Also, each particle has its own velocity $w^\alpha_A(\tau) \equiv dx^\alpha_A/d\tau_A$, and kinetic momentum p^α_A We shall denote $p^\alpha_A = m_A w^\alpha_A$ where $m_A = \mathring{m}_A$, $w^\alpha_A w_{A\alpha} = -1$ for non-zero rest mass particles and $m_A = 1$, $w^\alpha_A w_{A\alpha} = 0$ for zero rest mass particles. We shall refer to non-zero rest mass particles as massive particles; also, we shall refer to zero rest mass particles as massless particles.

At each space-time point x^α we construct a four-dimensional momentum space, whose co-ordinates are the components of momentum p^α_A, and which is a subspace of the eight-dimensional phase space (x^α, p^α_A) for species A. Not all of this momentum space is accessible to a particle. The condition $p^\alpha_A p_{A\alpha} = -\mathring{m}^2_A$ confines the particle to the surface of a pseudo-sphere in the momentum space. We say that the particle is on-shell.

We choose, at x^α, an arbitrary space-like element of three volume $d\Sigma$ with surface normal n^α. The element of volume of the accessible momentum space dV_A is the four dimensional volume element $\sqrt{-g}\, d^4 p^\alpha_A$ multiplied by the on-shell condition $\delta(p^\alpha_A p_{A\alpha} + \mathring{m}^2_A)\, \theta(p^4_A)$ [1].

The invariant distribution function for species A, $N_A(x^\alpha, p^\alpha_A)$ is defined by stating that [11] the number of world lines of particles of species A which cross $d\Sigma$ in the positive sense of the unit normal with momenta in the range of dV_A is given by

$$N_A(x^\alpha,p_A^\alpha)(n_\mu w_A^\mu)(n_\nu n^\nu)\,d\Sigma\, dV_A \quad . \tag{2.1}$$

We postulate that the behaviour of $N_A(x^\alpha,p_A^\alpha)$, and hence of the gas, is described by an equation of continuity in phase space with sources and sinks:

$$(N_A w_A^\lambda)||\lambda + \frac{\partial}{\partial p_A^\lambda}\left(N_A \frac{\delta p_A^\lambda}{\delta\tau}\right) = D_{coll} N_A \quad . \tag{2.2}$$

Here N_A is the abbreviated form for the distribution function, $||$ means covariant differentiation holding momentum fixed by parallel propagation, $\partial/\partial p_A^\lambda$ means differentiation with respect to momentum holding position fixed, and $\delta p_A^\lambda/\delta\tau$ is the rate of change of momentum along a particle's world line. Equation (2.2) may be regarded in a different manner. The left hand side is just the Liouville operator acting on the distribution function, and describes the rate of change of N_A along the streamlines in phase space [13]. In the absence of collisions this rate of change is zero. The right hand side is therefore a correction to the Liouville equation to account for collisions. Equation (2.2) is our Boltzmann equation.

Unless otherwise stated, we shall assume that there are no external fields other than gravity, so that the rate of change of momentum between collisions is zero, that is $\delta p_A^\lambda/\delta\tau = 0$.

Consider two particles of species A and B which undergo a collision. The incoming particles have momenta p^α_A and p^α_B, and the outgoing particles have momenta $\overset{*}{p}{}^\alpha_A$ and $\overset{*}{p}{}^\alpha_B$. The collision conserves momentum: $\overset{*}{p}{}^\alpha_A + \overset{*}{p}{}^\alpha_B = \overset{*}{p}{}^\alpha_A + \overset{*}{p}{}^\alpha_B$. The transition probability for this collision is denoted by $W(p_A,p_B|\overset{*}{p}_A,\overset{*}{p}_B)$. This transition probability is a scalar funtion of $p_A{}^\alpha$, $p_B{}^\alpha$, $\overset{*}{p}_A{}^\alpha$, $\overset{*}{p}_B{}^\alpha$, and has the property that [40]

$$W(p_A,p_B|\overset{*}{p}_A,\overset{*}{p}_B) = W(p_B,p_A|\overset{*}{p}_B,\overset{*}{p}_A) \ . \tag{2.3}$$

We shall also assume that the transition probability satisfies the bilateral normalization property which appears here in a form suggested by Weinberg [42]:

$$\begin{aligned}\int W(p_A,p_B|\overset{*}{p}_A,\overset{*}{p}_B)\overset{*}{\Delta}_A\overset{*}{\Delta}_B d\overset{*}{V}_A d\overset{*}{V}_B \\ = \int W(\overset{*}{p}_A,\overset{*}{p}_B|p_A,p_B)\overset{*}{\Delta}_A\overset{*}{\Delta}_B dV_A dV_B \ .\end{aligned} \tag{2.4}$$

In equation (2.4) we have $\Delta_A \equiv g_A+\varepsilon_A N_A$ with $g_A=2$ for zero rest mass particles, $g_A=2s_A+1$ for non-zero rest mass particles, and $g_A=1$ for classical particles; $\varepsilon_A=1$ for bosons, $\varepsilon_A=-1$ for fermions, and $\varepsilon_A=0$ for classical particles. For classical particles, equation (2.4) reduces to the form of the bilateral normalization as stated by de Groot et. al. [11] and can be derived from the unitarity of the scattering matrix in quantum mechanics [40]. We note that equation (2.4) is more general than the assumption of detailed balancing since the latter immediately implies the former.

We can now write out the source and sink terms for the Boltzmann equation. Unless otherwise stated, we shall only consider binary collisions, which automatically preserve the number of particles of each species. The source terms are the number of particles created in a momentum cell by collisions; and the sink terms are the number of particles destroyed by collisions in that cell. The difference between the two terms (source minus sink) is $D_{\text{coll}}N_A$:

$$\begin{aligned} D_{\text{coll}}N_A = &\sum_B \int N_A^* N_B^* W(p_A^*, p_B^* | p_A, p_B) \Delta_A^* \Delta_B^* dV_B dV_B^* dV_A^* \\ &-\sum_B \int N_A N_B W(p_A, p_B | p_A^*, p_B^*) \Delta_A^* \Delta_B^* dV_B dV_B^* dV_A^* \quad . \end{aligned} \tag{2.5}$$

The Δ terms in equation (2.5) account for the Fermi exclusion and stimulated emission quantum effects.

B. The Conservation Equations

A master balance equation can now be derived from equation (2.2). We let $\xi_A \equiv \Psi_A d\Phi_A / dN_A$ where Ψ_A is any tensor function of position or momentum and Φ_A is a function of N_A alone. We multiply equation (2.2) by ξ_A and integrate over the accessible momentum space:

$$\int \Psi_A \frac{d\Phi_A}{dN_A} \frac{\delta N_A}{\delta \tau} dV_A = \int \xi_A D_{\text{coll}} N_A dV_A \quad . \tag{2.6}$$

After some algebra, we obtain our master balance equation

$$\left\{\int \Phi_A \Psi_A w_A^{\alpha} dV_A\right\}_{|\alpha} = \int \Phi_A \frac{\delta\Psi_A}{\delta\tau}\, dV_A + \int \xi_A D_{coll} N_A dV_A \quad ; \tag{2.7}$$

where the single stroke on the left hand side means covariant differentiation.

The expanded form of the last term in equation (2.7) is given by

$$\begin{aligned}\int \xi_A D_{coll} N_A dV_A = {} & \sum_B \int \xi_A \overset{*}{N}_A \overset{*}{N}_B W(\overset{*}{p}_A,\overset{*}{p}_B|p_A,p_B)\Delta_A\Delta_B d^4V \\ & -\sum_B \int \xi_A N_A N_B W(p_A,p_B|\overset{*}{p}_A,\overset{*}{p}_B)\overset{*}{\Delta}_A\overset{*}{\Delta}_B d^4V \quad ;\end{aligned} \tag{2.8}$$

where $d^4V \equiv dV_A dV_B d\overset{*}{V}_A d\overset{*}{V}_B$.

When we relabel the variables in the first term in equation (2.8) we obtain the following expression:

$$\int \xi_A D_{coll} N_A dV_A = \sum_B \int (\overset{*}{\xi}_A - \xi_A) N_A N_B \overset{*}{\Delta}_A \overset{*}{\Delta}_B W(p_A,p_B|\overset{*}{p}_A,\overset{*}{p}_B) d^4V \ . \tag{2.9}$$

From equation (2.9) we see that if $\overset{*}{\xi}_A = \xi_A$, that is, ξ_A is a collisional invariant, then the right hand side of equation (2.9) is zero. If ξ_A is not a collisional invariant, we sum over all species so that equation (2.7) becomes

$$\sum_A\left\{\int \Phi_A \Psi_A w_A^{\alpha} dV_A\right\} = \sum_A \int \Phi_A \frac{\delta\Psi_A}{\delta\tau} dV_A + \sum_A \int \xi_A D_{coll} N_A dV_A \ . \tag{2.10}$$

The last term in equation (2.10) becomes, via relabelling of variables in equation (2.8) and using equation (2.3),

$$\sum_A \int \xi_A D_{coll} N_A dV_A = \frac{1}{2} \sum\sum_{AB} \int [\xi] N_A N_B \Delta^*_A \Delta^*_B W_{AB} d^4 V \quad ; \qquad (2.11)$$

where $[\xi] \equiv \xi^*_A + \xi^*_B - \xi_A - \xi_B$. If is a summational invariant of the collision, then $[\xi] = 0$.

The familiar conservation laws are now just particular cases of equations (2.7) and (2.10). We shall discuss each of these in turn.

The number flux N^α_A is defined by

$$N^\alpha_A \equiv \int N_A w^\alpha_A \, dV_A \quad . \qquad (2.12)$$

With $\Phi_A = N_A$ and $\Psi_A = 1$, equations (2.7) and (2.9) give us the number flux conservation equation:

$$N^\alpha_{A|\alpha} = 0 \quad . \qquad (2.13)$$

We note that this result depends heavily upon our decision to consider only elastic binary collisions.

The mass flux M^α_A is defined by

$$M^\alpha_A \equiv \int N_A p^\alpha_A \, dV_A \quad . \qquad (2.14)$$

We let $\Phi_A = N_A$ and $\Psi_A = m_A$; consequently equations (2.7) and

(2.9) give us the mass flux conservation equation:

$$M^{\alpha}_{A|\alpha} = 0 \quad . \tag{2.15}$$

The energy-momentum tensor of particle species A is defined by

$$T^{\alpha\beta}_{A} \equiv \int N_A p^{\alpha}_{A} w^{\beta}_{A} \, dV_A \quad ; \tag{2.16}$$

and we define the total energy-momentum tensor by

$$T^{\alpha\beta} \equiv \sum_{A} T^{\alpha\beta}_{A} \quad . \tag{2.17}$$

Noting that $[p_A] = 0$ for collisions we select $\Phi_A = N_A$ and $\Psi_A = p^{\alpha}_{A}$. Then equations (2.10) and (2.11) give us the energy-momentum conservation equation:

$$T^{\alpha\beta}{}_{|\beta} = 0 \quad . \tag{2.18}$$

In contrast to this, equation (2.7) gives us the following result:

$$T^{\alpha\beta}_{A\,|\beta} = \int p^{\alpha}_{A} \, D_{coll} N_A \, dV_A \quad . \tag{2.19}$$

A useful physical tensor denoted by $U^{\alpha\beta\gamma}_{A}$, and which we shall call the double-momentum flux, is defined by

$$U_A^{\alpha\beta\gamma} \equiv \int N_A p_A^\alpha w_A^\beta w_A^\gamma \, dV_A \quad . \tag{2.20}$$

With the choices $\Phi_A = N_A$ and $\Psi_A = p_A^\alpha w_A^\beta$, equation (2.7) gives us the following result:

$$U_A^{\alpha\beta\lambda}{}_{|\lambda} = \int p_A^\alpha w_A^\beta D_{coll} N_A \, dV_A \quad . \tag{2.21}$$

The entropy flux for species A is defined by

$$S_A^\alpha \equiv -k \int \{N_A \ln(N_A/g_A) - \varepsilon_A \Delta_A \ln(\Delta_A/g_A) + (\varepsilon_A^2 - 1) N_A\} w_A^\alpha dV_A \quad ; \tag{2.22}$$

and the total entropy flux is defined by

$$S^\alpha \equiv \sum_A S_A^\alpha \quad ; \tag{2.23}$$

where k is Boltzmann's constant. The first two terms in braces in equation (2.22) together comprise the familiar definition of entropy for fermions and bosons as reported by Nordheim [29]. The physical justification for these terms is that we must not only count particles but "holes" when we calculate the entropy [36]. The third term in braces in equation (2.22) is zero for fermions or bosons but for classical particles effectively subtracts the number flux from the usual definition of the entropy flux. Since the number flux is conserved, this term adds nothing to the entropy production $S^{\alpha}_{A|\alpha}$. This term is also useful mathematically because when we choose

$$\Phi_A = N_A \ln(N_A/g_A) - \varepsilon_A \Delta_A \ln(\Delta_A/g_A) + (\varepsilon_A^2-1)N_A \quad , \qquad (2.24)$$

we obtain $d\Phi_A/dN_A = \ln(N_A/\Delta_A)$ for all types of particles. Then, with $\Psi_A = -k$, equations (2.10) and (2.11) give us the entropy production equation:

$$S^{\alpha}{}_{|\alpha} = -k \sum_{AB}\sum \frac{1}{2} \int [\ln(N/\Delta)] N_A N_B \overset{*}{\Delta}_A \overset{*}{\Delta}_B W_{AB} d^4V \quad . \qquad (2.25)$$

A Boltzmann H theorem can now be derived. We multiply equation (2.4) by $N_A N_B$ and integrate over both momentum variables. When we relabel some of our variables in the resulting expression, we obtain the following result:

$$\frac{1}{2} \int (\Xi - 1) N_A N_B \overset{*}{\Delta}_A \overset{*}{\Delta}_B W_{AB} d^4V = 0 \quad ; \qquad (2.26)$$

where we have set $\Xi \equiv (\overset{*}{N}_A \overset{*}{N}_B/\overset{*}{\Delta}_A \overset{*}{\Delta}_B) \div (N_A N_B/\overset{*}{\Delta}_A \overset{*}{\Delta}_B)$. We now add equation (2.26) to equation (2.25); consequently we obtain

$$S^{\alpha}{}_{|\alpha} = -\frac{k}{2} \sum_{AB}\sum \int \{\ln(\Xi) - \Xi + 1\} N_A N_B \overset{*}{\Delta}_A \overset{*}{\Delta}_B W_{AB} d^4V \quad . \qquad (2.27)$$

We note that $N_A \geq 0$. Also, $\Delta_A \geq 0$ is required by the exclusion principle for fermions, and is obvious for classical particles and bosons [12]. Therefore $\Xi \geq 0$ and hence $\ln(\Xi) - \Xi + 1 \leq 0$. Consequently, equation (2.27) tells us that

$S^{\alpha}{}_{|\alpha} \geq 0$ which is Boltzmann's H theorem. When the entropy production is zero, $S^{\alpha}{}_{|\alpha} = 0$, we say that the gas is in equilibrium.

C. General Decompositions

Let us consider an arbitrary time-like vector field $u^{\alpha}(x^{\mu})$, $g_{\alpha\beta}u^{\alpha}u^{\beta} = -1$. We can decompose the covariant derivative of u^{α} by [31]

$$u_{\alpha|\beta} = -\dot{u}_{\alpha}u_{\beta} + \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{\theta}{3}\Delta_{\alpha\beta} \ , \tag{2.28}$$

where $\Delta_{\alpha\beta} \equiv g_{\alpha\beta} + u_{\alpha}u_{\beta}$, $\theta \equiv u^{\lambda}{}_{|\lambda}$ is the volume expansion, $\sigma_{\alpha\beta} \equiv (\Delta_{\alpha}{}^{\lambda}\Delta_{\beta}{}^{\tau} - \frac{1}{3}\Delta_{\alpha\beta}\Delta^{\lambda\tau})u_{(\alpha|\beta)}$ is the shear, $\omega_{\alpha\beta} \equiv \Delta_{\alpha}{}^{\lambda}\Delta_{\beta}{}^{\tau}u_{[\alpha|\beta]}$ is the vorticity, and $\dot{u}_{\alpha} \equiv u_{\alpha|\lambda}u^{\lambda}$ is the acceleration.

In general, we can decompose the mass flux M^{α}_{A} with respect to u^{α} by

$$M^{\alpha}_{A} = n_{A}u^{\alpha} + j^{\alpha}_{A} \ ; \tag{2.29}$$

$$n_{A} \equiv -u_{\lambda}M^{\lambda}_{A} \ , \quad j^{\alpha}_{A} \equiv \Delta^{\alpha\lambda}M_{A\lambda} \ . \tag{2.30}$$

The quantity n_{A} is called the mass density and the vector $j_{A}{}^{\alpha}$ is called the particle drift.

Similarily, decompostion of the energy-momentum tensor with respect to u^{α} gives us that

$$
\begin{aligned}
T_A^{\mu\nu} &= \rho_A u^\mu u^\nu + \tilde{P}_A \Delta^{\mu\nu} + h_A^\mu u^\nu + h_A^\nu u^\mu + \pi_A^{\mu\nu} \; ; \\
\rho_A &\equiv u_\mu u_\nu T_A^{\mu\nu} \; ; \; \tilde{P}_A \equiv \frac{1}{3} \Delta_{\mu\nu} T_A^{\mu\nu} \; ; \qquad (2.31) \\
h_A^\alpha &\equiv -\Delta^\alpha{}_\lambda u_\tau T_A^{\lambda\tau} \; ; \; \pi_A^{\mu\nu} \equiv (\Delta^\mu{}_\lambda \Delta^\nu{}_\tau - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\lambda\tau}) T_A^{\lambda\tau} \; .
\end{aligned}
$$

The energy density is ρ_A , the momentum flux is h_A^α , the viscous stresses are $\pi_A^{\alpha\beta}$, and we shall call $\tilde{P}_A$ the bulk pressure. This quantity is the thermodynamic pressure P_A only in equilibrium [14].

The decomposition of the mass flux and the energy-momentum tensor of species A presented above are convenient for the analysis of the equilibrium and non-equilibrium states of the gas. We shall consider equilibrium first; it will be discussed in chapter III.

III. Equilibrium

In kinetic theory, equilibrium maintained via collisions has a a special role; this role arises because the distribution function may be specified exactly. The specification of the equilibrium distribution function allows us to calculate the mass flux, the energy-momentum tensor, etc. exactly. These quantities and their balance equations describe the equilibrium behaviour of the gas and provide a foundation for the investigation of the physics of a gas in a state close to equilibrium.

In this chapter we shall discuss equilibrium. We shall first specify the equilibrium distribution function and define the chemical potentials and the temperature. Secondly, we shall examine the restrictions on the gradients of the thermal potentials and the temperature imposed by the Boltzmann equation itself. We shall define some standard integrals of the equilibrium distribution function which allow us to calculate the mass flux, the energy-momentum tensor etc., immediately. This analysis provides us with the functional forms for the mass densities, energy densities, partial pressures, and entropy in terms of the thermal potentials and the temperature. Consequently we may obtain a Gibbs' relation in equilibrium. Finally we obtain some expressions for the heat capacities at constant pressure and volume.

A. The Equilibrium Distribution Function

We have defined equilibrium to be that state for which the entropy production is zero. Let us denote the equilibrium situation by a superscript 0. When we examine equation (2.27) we note that zero entropy production requires that $[\ln(\overset{\circ}{N}/\overset{\circ}{\Delta})] = 0$. Thus $\ln(\overset{\circ}{N}/\overset{\circ}{\Delta})$ is a collisional invariant. The most general form for $\ln(\overset{\circ}{N}_A/\overset{\circ}{\Delta}_A)$ is given by [33]

$$\ln(\overset{\circ}{N}_A/\overset{\circ}{\Delta}_A) = \alpha_A(x) + \tilde{\beta}^\lambda(x)p_{A\lambda} \quad . \tag{3.1}$$

Equation (3.1) now implies that

$$\overset{\circ}{N}_A = \frac{g_A}{\exp(-\alpha_A-\tilde{\beta}^\lambda p_{A\lambda})-\varepsilon_A} \;;\; \overset{\circ}{\Delta}_A = \frac{g_A\exp(-\alpha_A-\tilde{\beta}^\lambda p_{A\lambda})}{\exp(-\alpha_A-\tilde{\beta}^\lambda p_{A\lambda})-\varepsilon_A} \quad . \tag{3.2}$$

The coefficients α_A and β_A have physical interpretations. The coefficient α_A is the relativistic chemical potential and is related to the classical chemical potential μ_A by [14]

$$\alpha_A = (m_A/T)(\,1 + \mu_A/c^2\,) \text{ , (c.g.s. units).} \tag{3.3}$$

To avoid confusion between the two types of chemical potential we shall refer to α_A as the thermal potential of species A.

We define a temperature T(x) by $\tilde{\beta} \equiv 1/kT$ where $\tilde{\beta} \equiv (-\tilde{\beta}^{\lambda}\tilde{\beta}_{\lambda})^{.5}$. We must have that $\tilde{\beta}^{\lambda}$ be time-like so that integrals of the distribution function converge. We also define a unit flow vector u^{α} by $u^{\alpha} \equiv \tilde{\beta}^{\alpha}/\tilde{\beta}$; hence $\tilde{\beta} = \tilde{\beta}u^{\alpha}$. The unit flow vector u^{α} may also be considered to be the velocity of an observer moving with the fluid (comoving observer). For convenience we also define $\beta_A \equiv m_A\tilde{\beta}$ and $\beta_A^{\alpha} \equiv m_A\tilde{\beta}^{\alpha}$.

In equilibrium, the form of the distribution function implies that $D_{coll}N_A = 0$. The Boltzmann equation now places restrictions on α_A and $\tilde{\beta}$:

$$\alpha_{A|\lambda}w_A^{\lambda} + \tilde{\beta}_{\mu|\lambda}p_A{}^{\mu}w_A{}^{\lambda} = 0 \quad . \tag{3.4}$$

This must hold true for arbitrary time-like or null momenta. Hence $\alpha_{A|\lambda}=0$ for all species. If the gas consists of only massless particles we conclude that $\tilde{\beta}^{\alpha}$ is conformal Killing: $\tilde{\beta}_{(\mu|\nu)} = \frac{1}{2}g_{\mu\nu}\tilde{\beta}^{\lambda}{}_{|\lambda}$. However, if at least one component of our gas consists of massive particles, then $\tilde{\beta}^{\alpha}$ is a Killing vector: $\tilde{\beta}_{(\mu|\nu)}=0$. Thus the space-time is stationary.

Since $\tilde{\beta}_{\alpha}$ is a Killing vector we conclude, after some algebraic manipulation that $\tilde{\beta}_{|\mu} = \tilde{\beta}\dot{u}_{\mu}$, $\sigma_{\mu\nu} = 0$, and $\theta=0$. Hence the gas is also shear free and expansionless.

B. Standard Integrals and Functions

In equilibrium, all moments of the distribution function can be expanded in terms of a set of standard integrals of the equilibrium distribution function. These standard integrals are defined by

$$I_A^{\alpha_1\ldots\alpha_n} \equiv \frac{1}{m_A^{\,n-1}} \int \overset{\circ}{N}_A p_A^{\alpha_1}\ldots p_A^{\alpha_n}\, dV_A \;;$$

$$J_A^{\alpha_1\ldots\alpha_n} \equiv \frac{1}{g_A m_A^{\,n-1}} \int \overset{\circ}{N}_A \overset{\circ}{\Delta}_A p_A^{\alpha_1}\ldots p_A^{\alpha_n}\, dV_A \;. \tag{3.5}$$

These integrals single out only one direction in space-time, u_α , so that they must be constructed out of scalars, the metric tensor, and the flow vector u^α . Therefore, we define the following quantities [13]:

$$\Pi_{(q)}^{\alpha_1\ldots\alpha_n} \equiv \Delta^{(\alpha_1\alpha_2}\ldots\Delta^{\alpha_{2q-1}\alpha_{2q}}u^{\alpha_{2q+1}}\ldots u^{\alpha_n)} \;; \tag{3.6}$$

$$a_{nq} \equiv \binom{n}{2q}(2q-1)!! \;; \tag{3.7}$$

where $2q \leq n$. We then have the following orthogonality relation:

$$\Pi_{(q')}{}^{\alpha_1\ldots\alpha_n}\, \Pi_{(q)\alpha_1\ldots\alpha_n} = (-1)^n(2q+1)\delta_{qq'} \div \binom{n}{2q} \;. \tag{3.8}$$

Expanding equation (3.5) gives us

$$I_A^{\alpha_1\ldots\alpha_n} = \sum_{q=0}^{n/2} a_{nq} I_{Anq} \Pi_{(q)}{}^{\alpha_1\ldots\alpha_n} \;; \tag{3.9}$$

$$J_A^{\alpha_1\ldots\alpha_n} = \sum_{q=0}^{n/2} a_{nq} J_{Anq} \Pi_{(q)}{}^{\alpha_1\ldots\alpha_n} \;. \tag{3.10}$$

Contracting equations (3.9) and (3.10) with (3.6) and employing the orthogonality relations (3.8) produces

$$I_{Anq} \equiv \frac{(-1)^n}{(2q+1)!!} I_A{}^{\alpha_1 \cdots \alpha_n} \Pi_{(q)\alpha_1 \cdots \alpha_n} \; ; \tag{3.11}$$

$$J_{Anq} \equiv \frac{(-1)^n}{(2q+1)!!} J_A{}^{\alpha_1 \cdots \alpha_n} \Pi_{(q)\alpha_1 \cdots \alpha_n} \; . \tag{3.12}$$

Integral representations for these coefficients are deduced in Appendix A. Appendix A also includes general differential and recursion relations between these coefficients which are extremely useful.

C. The Physics of the Equilibrium

We can now examine the equilibrium form of the moments introduced in chapter II. In equilibrium, $N_A = \overset{\circ}{N}_A$, so that for the mass flux we obtain $M_A^\alpha = I_A^\alpha = I_{A10} u^\alpha$. When we apply the decomposition (2.29) with respect to the flow vector u_α , we conclude that $n_A = I_{A10}$ and $j_A^\alpha = 0$; furthermore, the number density becomes $\tilde{n}_A = I_{A10}/m_A$.

Similarily, for the energy-momentum tensor we obtain $T_A^{\mu\nu} = I_A^{\mu\nu} = I_{A20} u^\mu u^\nu + I_{A21} \Delta^{\mu\nu}$. Comparing this result with the decomposition (2.31) with respect to the flow vector u^α, we obtain $\rho_A = I_{A20}$, $\tilde{P}_A = P_A = I_{A21}$, $h_A^\alpha = 0$, and $\pi_A^{\alpha\beta} = 0$. We define the bulk stress to be $\pi_A/3 \equiv \tilde{P}_A - I_{A21}$ so that in equilibrium $\pi_A = 0$.

In equilibrium $\Phi_A(N_A)$ as given by equation (2.24) is

$$\Phi(\overset{\circ}{N}_A) = \overset{\circ}{N}_A(\alpha_A + \tilde{\beta}^{\lambda} p_{A\lambda}) + (\varepsilon_A^2 - 1)\overset{\circ}{N}_A + g_A \varepsilon_A \ln(1 - \varepsilon_A e^{\alpha_A + \tilde{\beta}^{\lambda} p_{A\lambda}}) . \tag{3.13}$$

Employing this expression in equation (2.22) gives us the entropy flux in equilibrium:

$$\overset{\circ}{S}{}^{\alpha}_A = -k(\alpha_A + \varepsilon_A^2 - 1)\overset{\circ}{N}{}^{\alpha}_A - k\tilde{\beta}_{\lambda}\overset{\circ}{T}{}^{\lambda\alpha}_A + \hat{S}^{\alpha}_A \quad ; \tag{3.14}$$

$$\hat{S}^{\alpha}_A = \hat{S}_A u^{\alpha} \ , \quad \hat{S}_A \equiv -k g_A \varepsilon_A \int \ln\left\{1 - \varepsilon_A \exp(\alpha_A + \tilde{\beta}_{\lambda} w^{\lambda}_A)\right\} dV_A \ . \tag{3.15}$$

Israel and Stewart [16] have shown that $\hat{S}_A = k\varepsilon_A^2 \tilde{\beta}\, I_{A21}$. The entropy is defined by $S_A \equiv -u_{\alpha} S^{\alpha}_A$. Noting that $(\varepsilon_A^2 - 1)(\tilde{n}_A - \tilde{\beta}P_A) = 0$ for all species we then have that

$$S_A = -k\left\{ \alpha_A \tilde{n}_A - \tilde{\beta}(\rho_A + P_A) \right\} . \tag{3.16}$$

Defining $\Theta_A \equiv k\alpha_A / m_A$ we deduce that

$$TS_A + T\Theta_A n_A - \rho_A = P_A \ . \tag{3.17}$$

When we differentiate equation (3.16) and apply relationship (A19) to P_A (i.e. I_{A21}), we obtain the following result:

$$TdS_A = d\rho_A - T\Theta_A dn_A \quad . \qquad (3.18)$$

The differentials appearing in this and subsequent equations in this chapter are constrained to virtual displacements between equilibrium states only. Multiplying equation (3.18) by the arbitrary parameter $\tilde{V}$ and using (3.17) gives us the Gibbs relation

$$Td(S_A\tilde{V}) = d(\rho_A\tilde{V}) + P_A d\tilde{V} - T\Theta_A d(n_A\tilde{V}) \quad . \qquad (3.19)$$

When we perform a Legendre transformation $H_A \equiv (\rho_A+P_A)/\tilde{V}$ we obtain an alternative form of equation (3.19):

$$Td(S_A\tilde{V}) = dH_A - \tilde{V}dP_A - T\Theta_A d(n_A\tilde{V}) \quad . \qquad (3.20)$$

Let us now choose $\tilde{V}=1/n$ where $n=\sum_A n_A$. We then define the specific entropy per unit mass by $\sigma \equiv \sum_A S_A/n$, the enthalpy by $H \equiv \sum_A(\rho_A+P_A)/n$, the internal energy by $\bar{U} \equiv \sum_A \rho_A/n$, and $\bar{n}_A \equiv n_A/n$ which is the fractional proportion by mass of species A. Employing these definitions and summing equations (3.19) and (3.20) over all species A produces the following forms for the Gibbs relation:

$$\begin{aligned} Td\sigma &= dU + Pd\tilde{V} - T\sum_A \Theta_A d\bar{n}_A \\ &= dH - \tilde{V}dP - T\sum_A \Theta_A d\bar{n}_A \quad . \end{aligned} \qquad (3.21)$$

For constant chemical composition $d\bar{n}_A = 0$. This requires that

$$(d\alpha_A)_{\bar{n}_A} = -\frac{I_{A10}}{J_{A10}}\frac{d\tilde{V}}{\tilde{V}} - \frac{m_A J_{A20}}{J_{A10}}\frac{\tilde{\beta}}{T}\,dT \quad . \tag{3.22}$$

Consequently, by equation (A17) applied to J_{A31} we obtain

$$(dP_A)_{\bar{n}_A} = -\frac{J_{A21}I_{A10}}{J_{A10}}\frac{d\tilde{V}}{\tilde{V}} + \frac{\tilde{\beta}}{T}\frac{m_A J_{A21}}{J_{A10}}(J_{A10}\eta_A - J_{A20})\,dT \quad ; \tag{3.23}$$

where η_A is defined by equation (A22). Summing this equation over all species and rearranging the result gives us the relation

$$\frac{d\tilde{V}}{\tilde{V}} = -\kappa\,dP + \tilde{\alpha}\,dT \quad ; \tag{3.24}$$

where the isothermal compressibility is

$$\kappa \equiv -\frac{1}{\tilde{V}}\left(\frac{\partial\tilde{V}}{\partial P}\right)_T \; ; \; \frac{1}{\kappa} = \sum_A \frac{J_{A21}I_{A10}}{J_{A10}} \quad ; \tag{3.25}$$

and the coefficient of volume expansion is

$$\tilde{\alpha} \equiv \frac{1}{\tilde{V}}\left(\frac{\partial\tilde{V}}{\partial T}\right)_P = \kappa\frac{\tilde{\beta}}{T}\sum_A \frac{m_A J_{A21}}{J_{A10}}(J_{A10}\eta_A - J_{A20}) \quad . \tag{3.26}$$

Inserting equation (3.24) into (3.23) allows us to conclude that

$$\begin{aligned}(dP_A)_{\bar{n}_A} &= \kappa\frac{J_{A21}I_{A10}}{J_{A10}}\,dP \\ &\quad + \frac{\tilde{\beta}}{T}\left\{\frac{m_A J_{A21}}{J_{A10}}(J_{A10}\eta_A - J_{A20}) - \frac{J_{A21}I_{A10}}{J_{A10}}\frac{T\tilde{\alpha}}{\tilde{\beta}}\right\}dT \quad .\end{aligned} \tag{3.27}$$

For a gas consisting only of classical particles, $I_{Anq} = J_{Anq}$; so we obtain, after some algebra, that $(dP_A)_{\bar{n}_A,P} = 0$. Hence, constant pressure implies that the partial pressures are constant for a gas of classical particles.

When we employ equation (A19) for I_{A20} along with equation (3.22) we obtain

$$(d\rho)_{\bar{n}_A} = -\sum_A \frac{J_{A20} I_{A10}}{J_{A10}} \frac{d\tilde{V}}{\tilde{V}} + \frac{\tilde{\beta}}{T} \left\{ \sum_A \frac{m_A D_{A20}}{J_{A10}} \right\} dT \quad ; \qquad (3.28)$$

where D_{A20} is defined by equation (A21). Inserting equation (3.24) into (3.28) gives us an alternative form of equation (3.28):

$$\begin{aligned}(d\rho)_{\bar{n}_A} &= \kappa \sum_A \frac{J_{A20} I_{A10}}{J_{A10}} dP \\ &+ \left\{ \frac{\tilde{\beta}}{T} \sum_A \frac{m_A D_{A20}}{J_{A10}} - \tilde{\alpha} \sum_A \frac{J_{A20} I_{A10}}{J_{A10}} \right\} dT \quad .\end{aligned} \qquad (3.29)$$

The specific heat at constant volume is now defined and given by

$$C_{\tilde{V},\bar{n}_A} \equiv \left(\frac{\partial U}{\partial T}\right)_{V,\bar{n}_A} = \frac{1}{n} \frac{\tilde{\beta}}{T} \sum_A \frac{m_A D_{A20}}{J_{A10}} \quad . \qquad (3.30)$$

The specific heat at constant pressure is defined by

$$C_{P,\bar{n}_A} \equiv \left(\frac{\partial H}{\partial T}\right)_{P,\bar{n}_A} = \tilde{V}\left(\frac{\partial \rho}{\partial T}\right)_{P,\bar{n}_A} + (\rho + P)\left(\frac{\partial \tilde{V}}{\partial T}\right)_{P,\bar{n}_A} \quad . \qquad (3.31)$$

Via equations (3.29) and (3.26) we obtain

$$C_{P,\bar{n}_A} = C_{V,\bar{n}_A} + \frac{\tilde{\alpha}}{n} \left\{ \rho + P - \sum_A \frac{J_{A20} I_{A10}}{J_{A10}} \right\} \quad . \qquad (3.32)$$

The adiabatic index γ is defined by

$$\gamma \equiv \left(\frac{\partial \ln(P)}{\partial \ln(n)}\right)_{\sigma,\bar{n}_A} . \tag{3.33}$$

After some algebra we obtain the familiar result that

$$\gamma = \frac{1}{\kappa P} C_{P,\bar{n}_A} \div C_{V,\bar{n}_A} . \tag{3.34}$$

The calculations from equations (3.22) to (3.34) only apply to a system not containing photons. This is because, as the temperature increases, the number of photons in a fixed volume increases. Hence the proportion by mass of photons increases, that is, $d\bar{n}_A \neq 0$ for photons. Thus, for a gas of matter and photons, we apply the above calculations to the matter part of the gas and employ the procedure reported by Chandrasekhar [6] to compute specific heats.

We now have a number of useful formulae which enable us to examine the non-equilibrium case, which is treated in chapter IV.

IV. Non-equilibrium

The non-equilibrium state is much more complex than equilibrium and discussion of it will take the remainder of this thesis. In this chapter we wish to discuss the approach we will employ, namely, the fourteen moment relativistic Grad method.

We shall begin our analysis by specifying the distribution function in the fourteen moment approximation. This distribution function will then be employed to find the mass flux and energy-momentum tensor in terms of the deviations from equilibrium which appear in the distribution function. However, the mass flux and energy-momentum tensor may be decomposed in terms of mass density, energy density, heat flux, etc. Hence we can specify the relationship between these physical quantities and the deviations of the distribution function from equilibrium. This relationship is then used to find the structures of the double-momentum flux and the entropy flux in terms of thermodynamic functions and the physical quantities appearing in the mass flux and energy-momentum tensor. Our non-equilibrium solution permits us to compare the actual distribution function with an arbitrarily chosen nearby equilibrium distribution function. Thus, in the final section of this chapter we are motivated to show that our description of the gas is independent of this choice.

A. The Grad Fourteen Moment Method

The equilibrium state does not manifest some interesting physical phenomena such as shear viscosity, bulk viscosity, and heat flux. To describe such phenomena we must discuss the non-equilibrium state.

To examine non-equilibrium, we employ the relativistic Grad method of moments. We let

$$y_A(N_A) \equiv \ln(N_A/\Delta_A) = \ln(\overset{\circ}{N}_A/\overset{\circ}{\Delta}_A) + f_A \quad ; \tag{4.1}$$

where $\ln(\overset{\circ}{N}_A/\overset{\circ}{\Delta}_A)$ is given by equation (3.1) with $\tilde{\beta}^\lambda$ (common to all species) replaced by $\tilde{\beta}^\lambda_A$ (unique for each of the species). We also have $\tilde{\beta}^\lambda_A = \tilde{\beta}_A u^\lambda_A$, $u^\lambda_A u_{A\lambda} = -1$ and $\beta_A = m_A \tilde{\beta}_A$. The quantity f_A is assumed to be small compared to $\ln(\overset{\circ}{N}_A/\overset{\circ}{\Delta}_A)$ and is said to be first order (O_1) ; this informs us that we are "close" to equilibrium.

The relativistic Grad fourteen moment approximation is obtained by assuming that f_A is a quadratic function of momentum:

$$f_A = a_A(x) + \tilde{b}^\lambda_A(x) w_{A\lambda} + \tilde{c}^{\lambda\tau}_A(x) w_{A\lambda} w_{A\tau} \quad . \tag{4.2}$$

The quantities a_A , $\tilde{b}^\lambda_A$, $\tilde{c}^{\lambda\tau}_A$ are unknown first order variables which describe the deviation from equilibrium. We may assume that $\tilde{c}^{\lambda\tau}_A$ is trace free because a non-zero trace can be absorbed into a_A .

To facilitate computation, we define the following quantities:

$$b_A \equiv u_{A\lambda}\tilde{b}^{\lambda}_A \;;\quad b_{A\alpha} \equiv \Delta_{A\alpha\lambda}\tilde{b}^{\lambda}_A \;;$$

$$c_A \equiv u_{A\lambda}u_{A\tau}\tilde{c}^{\lambda\tau}_A \;;\quad c_{A\alpha} \equiv \Delta_{A\alpha\lambda}u_{A\tau}\tilde{c}^{\lambda\tau}_A \;; \qquad (4.3)$$

$$c_{A\alpha\beta} \equiv \Delta_{A\alpha\lambda}\Delta_{A\lambda\tau}\tilde{c}^{\lambda\tau}_A - \frac{1}{3}c_A\Delta_{A\alpha\beta} \,.$$

Note that b^{α}_A , c^{α}_A , and $c^{\alpha\beta}_A$ are all orthogonal to u^{α}_A . With these definitions we may rewrite equations (4.2) and (4.1) as

$$f_A = a_A + (-b_A u^{\alpha}_A + b^{\alpha}_A)w_{A\alpha}$$

$$+ \left\{ c_A(u^{\alpha}_A u^{\beta}_A + \frac{1}{3}\Delta^{\alpha\beta}_A) - (c^{\alpha}_A u^{\beta}_A + c^{\beta}_A u^{\alpha}_A) + c^{\alpha\beta}_A \right\} w_{A\alpha}w_{A\beta}; \qquad (4.4)$$

$$\ln(N_A/\Delta_A) = (\alpha_A + a_A) + [(\beta_A - b_A)u^{\alpha}_A + b^{\alpha}_A]w_{A\alpha}$$

$$+ \left\{ c_A(u^{\alpha}_A u^{\beta}_A + \frac{1}{3}\Delta^{\alpha\beta}_A) - (c^{\alpha}_A u^{\beta}_A + c^{\beta}_A u^{\alpha}_A) + c^{\alpha\beta}_A \right\} w_{A\alpha}w_{A\beta} \,. \qquad (4.5)$$

When we examine equation (4.5) we note that it is invariant under two classes of transformations of order one. The first class of transformations are frame changes:

$$u^{\alpha}_A = u'^{\alpha}_A + \delta u^{\alpha}_A \,,\quad u'_{A\alpha}\delta u^{\alpha}_A = 0_2 \;;$$

$$b_{A\alpha} = b'_{A\alpha} - \beta_A \delta u_{A\alpha} \,. \qquad (4.6)$$

The second class of transformations we shall call fitting changes and consists of two separate types:

$$\alpha_A = \alpha'_A + \delta\alpha_A \,,\quad a_A = a'_A - \delta\alpha_A \;; \qquad (4.7)$$

$$\beta_A = \beta'_A + \delta\beta_A \quad ; \quad b_A = b'_A + \delta\beta_A \quad . \qquad (4.8)$$

Since the individual species are close to a common equilibrium, we expect that all of the u^α_A's are close to a common u^α , that is $u^\alpha_A = u^\alpha + \delta^\alpha_A$ where δ^α_A is O_1 . We therefore employ the frame changes, equation (4.6), to adjust equations (4.4) and (4.5) so that we have a common u^α for all species. Once we have done this however, we still have the freedom of frame changes

$$u^\alpha = u'^\alpha + \delta u^\alpha \ , \ u'_\alpha \delta u^\alpha = O_2 \ ; \ b^\alpha_A = b'^\alpha_A - \beta_A \delta u^\alpha \quad , \qquad (4.9)$$

applied to all species.

These first order transformations do not carry any physical content. The specification of an equilibrium distribution function close to the actual distribution function is not unique. The first order transformations allow us to transform to a different choice of the equilibrium distribution function. Our choice of this function, which we "match" to the actual one, is therefore just a matter of mathematical convenience.

We define the deviation from equilibrium, δN_A , of the distribution function N_A by

$$N_A = \mathring{N}_A + \delta N_A \quad . \qquad (4.10)$$

Then, when we regard equation (4.1) as a Taylor series expansion of $\ln(N_A/\Delta_A)$ about the equilibrium value $\ln(\mathring{N}_A/\mathring{\Delta}_A)$ in terms of δN_A to first order, we have

$$\ln(N_A/\Delta_A) = \ln(\mathring{N}_A/\mathring{\Delta}_A) + \left.\frac{\partial y_A}{\partial N_A}\right]_{N_A=\mathring{N}_A} \delta N_A \quad . \tag{4.11}$$

Hence we can immediately identify δN_A :

$$\delta N_A = \frac{\mathring{N}_A \mathring{\Delta}_A}{g_A} f_A \quad . \tag{4.12}$$

Furthermore, we have for Δ_A the first order expansion

$$\Delta_A = \mathring{\Delta}_A + \varepsilon_A \delta N_A = \mathring{\Delta}_A + \frac{\varepsilon_A \mathring{N}_A \mathring{\Delta}_A}{g_A} f_A \quad . \tag{4.13}$$

B. The Non-equilibrium Tensor Structures

We now have all the information necessary to deduce the expressions for the mass flux, energy-momentum tensor, etc., to first order.

From the definition for mass flux, equation (2.14) and employing equations (4.10), (4.12), the detailed structure of f_A, and the definitions (3.5), we obtain the following intermediate result:

$$M_A^{\alpha} = I_A^{\alpha} + a_A J_A + \tilde{b}_A^{\lambda} J_{A\lambda}{}^{\alpha} + \tilde{c}_A^{\lambda\tau} J_{A\lambda\tau}{}^{\alpha} \quad . \tag{4.14}$$

Now, using the definitions (4.3) and the detailed structure of equations (3.9) and (3.10), and comparing the results with equation (2.29) allows us to identify

$$n_A = I_{A10} + \delta n_A \quad ; \tag{4.15}$$

$$j_A^{\alpha} = J_{A21} b_A^{\alpha} + 2J_{A31} c_A^{\alpha} \quad ; \tag{4.16}$$

where δn_A is given by

$$\delta n_A = J_{A10} a_A + J_{A20} b_A + (J_{A30} + J_{A31}) c_A \quad . \tag{4.17}$$

We may split the mass flux into zeroth and first order parts:

$$M_A^{\alpha} = \overset{\circ}{M}{}_A^{\alpha} + \delta M_A^{\alpha} \quad ; \tag{4.18}$$

where

$$\overset{\circ}{M}{}_A^{\alpha} = I_{A10} u^{\alpha} \quad ; \tag{4.19}$$

$$\delta M_A^{\alpha} = \delta n_A u^{\alpha} + j_A^{\alpha} \quad . \tag{4.20}$$

This split will facilitate computations later on.

In a mathematical treatment similar to that above we obtain for the energy-momentum tensor the following intermediate result:

$$T_A^{\alpha\beta} = I_A^{\alpha\beta} + a_A J_A^{\alpha\beta} + \bar{b}_A^{\lambda} J_{A\lambda}{}^{\alpha\beta} + \bar{c}_A^{\lambda\tau} J_{A\lambda\tau}{}^{\alpha\beta} \quad . \tag{4.21}$$

Expanding equation (4.21) via equations (3.9) and (3.10), and comparing the result with equation (2.31) allows us to obtain the following results:

$$\rho_A = I_{A20} + \delta\rho_A \quad ; \tag{4.22}$$

$$\delta\rho_A = J_{A20} a_A + J_{A30} b_A + (J_{A40} + J_{A41}) c_A \quad ; \tag{4.23}$$

$$\pi_A = 3J_{A21} a_A + 3J_{A31} b_A + (3J_{A41} + 5J_{A42}) c_A \quad ; \tag{4.24}$$

$$h_A^{\alpha} = J_{A31} b_A^{\alpha} + 2J_{A41} c_A^{\alpha} \quad ; \tag{4.25}$$

$$\pi_A^{\alpha\beta} = 2J_{A42} c_A^{\alpha\beta} \quad . \tag{4.26}$$

We split the energy-momentum tensor into zeroth and first order parts by letting

$$T_A^{\alpha\beta} = \overset{\circ}{T}{}_A^{\alpha\beta} + \delta T_A^{\alpha\beta} \quad ; \tag{4.27}$$

where

$$\overset{\circ}{T}{}_A^{\alpha\beta} = I_{A20}u^\alpha u^\beta + I_{A21}\Delta^{\alpha\beta} \; ; \tag{4.28}$$

$$\delta T_A^{\alpha\beta} = \delta\rho_A u^\alpha u^\beta + \frac{\pi_A}{3}\Delta^{\alpha\beta} + h_A^\alpha u^\beta + h_A^\beta u^\alpha + \pi_A^{\alpha\beta} \; . \tag{4.29}$$

Let us define the heat flux q_A^α by

$$q_A^\alpha \equiv h_A^\alpha - \eta_A J_A^\alpha \; . \tag{4.30}$$

This quantity is invariant to first order under frame changes, equation (4.9), (Israel and Stewart[16]). Then, from equations (4.16) and (4.25) we have

$$q_A^\alpha = 2J_{A21}\Lambda_A c_A^\alpha \; ; \tag{4.31}$$

where Λ_A is given by

$$\Lambda_A \equiv D_{A31}/J_{A31} = J_{A41}/J_{A21} - \eta_A^2 \; . \tag{4.32}$$

The physically measurable quantities in our theory are the mass density n_A, the energy density ρ_A, the bulk stress π_A, the momentum flux h_A^α, the heat flux q_A^α, and the viscous stresses $\pi_A^{\alpha\beta}$. If we specify the variables α_A and β_A for each species then equations (4.15) and (4.22) specify δn_A and $\delta\rho_A$. Conversely, if we specify δn_A and $\delta\rho_A$

then we can find α_A and β_A. We call these the fitting conditions. This justifies calling the transformations (4.7), (4.8) fitting changes. We choose to let our fitting conditions remain arbitrary so that our subsequent analysis remains independent of any choice of the fitting conditions.

C. The Solution for the Deviations from Equilibrium

For massive particles, equations (4.17), (4.23), (4.24), (4.25), (4.26), and (4.31) relate the fourteen unknown variables (a_A, b_A, c_A, b^{α}_A, c^{α}_A, $c^{\alpha\beta}_A$) to the fourteen known variables (δn_A, $\delta\rho_A$, π_A, h^{α}_A, q^{α}_A, $\pi^{\alpha\beta}_A$) . We now have a fourteen dimensional linear algebraic system of equations which is conveniently expressible as a six dimensional transformation:

$$\begin{bmatrix} \delta n_A \\ \delta\rho_A \\ \pi_A \\ h^{\alpha}_A \\ q^{\alpha}_A \\ \pi^{\alpha\beta}_A \end{bmatrix} = \begin{bmatrix} J_{A10} & J_{A20} & J_{A30}+J_{A31} & 0 & 0 & 0 \\ J_{A20} & J_{A30} & J_{A40}+J_{A41} & 0 & 0 & 0 \\ 3J_{A21} & 3J_{A31} & 3J_{A41}+5J_{A42} & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{A31} & 2J_{A41} & 0 \\ 0 & 0 & 0 & 0 & 2J_{A21}\Lambda_A & 0 \\ 0 & 0 & 0 & 0 & 0 & 2J_{A42} \end{bmatrix} \begin{bmatrix} a_A \\ b_A \\ c_A \\ b^{\alpha}_A \\ c^{\alpha}_A \\ c^{\alpha\beta}_A \end{bmatrix} . \qquad (4.33)$$

In the case of massless particles, the energy-momentum tensor is trace-free, that is $T^{\lambda}_{A\lambda} = 0$. This condition, via equation (2.31), requires $\rho_A = 3\tilde{P}_A$ and hence $\delta\rho_A = \pi_A$.

Therefore, we only have thirteen equations relating the fourteen variables (a_A, b_A, c_A, b_A^α, c_A^α, $c_A^{\alpha\beta}$) to the thirteen variables (δn_A, $\delta\rho_A$, h_A^α, q_A^α, $\pi_A^{\alpha\beta}$). Hence the problem is not solvable with the given information. We shall have to employ a different approach to the massless case. Hence, we shall defer discussion of massless particles to chapter VI. It will now be understood that, in what follows, we are dealing solely with massive particles for all species.

Let us denote the inverse matrix of the matrix appearing in equation (4.33) by (Ω) with matrix elements Ω_{Aij} (i,j = 1→6). Then the inverse relation to equation (4.33) is given by

$$\begin{bmatrix} a_A \\ b_A \\ c_A \\ b_A^\alpha \\ c_A^\alpha \\ c_A^{\alpha\beta} \end{bmatrix} = \begin{bmatrix} \Omega_{A11} & \Omega_{A12} & \Omega_{A13} & 0 & 0 & 0 \\ \Omega_{A21} & \Omega_{A22} & \Omega_{A23} & 0 & 0 & 0 \\ \Omega_{A31} & \Omega_{A32} & \Omega_{A33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_{A44} & \Omega_{A45} & 0 \\ 0 & 0 & 0 & 0 & \Omega_{A55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega_{A66} \end{bmatrix} \begin{bmatrix} \delta n_A \\ \delta\rho_A \\ \pi_A \\ h_A^\alpha \\ q_A^\alpha \\ \pi_A^{\alpha\beta} \end{bmatrix} . \qquad (4.34)$$

Let us define Ω_A by

$$\Omega_A \equiv \frac{3J_{A41}(J_{A10}J_{A31}-J_{A20}J_{A31})+3J_{A31}(J_{A21}J_{A30}-J_{A20}J_{A31})}{J_{A42}D_{A20}} - 5. \qquad (4.35)$$

Then the coefficients Ω_{Aij} appearing in equation (4.34) are given by

$$\Omega_{A11} = \frac{3(J_{A31}J_{A40}-J_{A30}J_{A41}) + 3J_{A31}J_{A41} - 5J_{A30}J_{A42}}{4J_{A42}D_{A20}\Omega_{A}} \quad ; \tag{4.36}$$

$$\Omega_{A12} = \frac{J_{A20}(3J_{A41}+5J_{A42}) - 3J_{A31}(J_{A30}+J_{A31})}{4J_{A42}D_{A20}\Omega_{A}} \quad ; \tag{4.37}$$

$$\Omega_{A13} = \frac{J_{A30}(J_{A30}+J_{A31}) - J_{A20}(J_{A40}+J_{A41})}{4J_{A42}D_{A20}\Omega_{A}} \quad ; \tag{4.38}$$

$$\Omega_{A21} = \frac{J_{A20}(3J_{A41}+5J_{A42}) - 3J_{A21}(J_{A40}+J_{A41})}{4J_{A42}D_{A20}\Omega_{A}} \quad ; \tag{4,39}$$

$$\Omega_{A22} = \frac{3J_{A21}(J_{A30}+J_{A31}) - J_{A10}(3J_{A41}+5J_{A42})}{4J_{A42}D_{A20}\Omega_{A}} \quad ; \tag{4.40}$$

$$\Omega_{A23} = \frac{J_{A10}(J_{A40}+J_{A41}) - J_{A20}(J_{A30}+J_{A31})}{4J_{A42}D_{A20}\Omega_{A}} \quad ; \tag{4.41}$$

$$\Omega_{A31} = \frac{3(J_{A21}J_{A30}-J_{A20}J_{A31})}{4J_{A42}D_{A20}\Omega_{A}} \quad ; \tag{4.42}$$

$$\Omega_{A32} = \frac{3(J_{A10}J_{A31}-J_{A20}J_{A21})}{4J_{A42}D_{A20}\Omega_{A}} \quad ; \tag{4.43}$$

$$\Omega_{A33} = -1/(4J_{A42}\Omega_{A}) \quad ; \tag{4.44}$$

$$\Omega_{A44} = 1/J_{A31} \quad ; \qquad (4.45)$$

$$\Omega_{A45} = - J_{A41}/(J_{A21}J_{A31}\Lambda_A) \quad ; \qquad (4.46)$$

$$\Omega_{A55} = 1/(2J_{A21}\Lambda_A) \quad ; \qquad (4.47)$$

$$\Omega_{A66} = 1/(2J_{A42}) \quad . \qquad (4.48)$$

The structures of the mass flux and energy-momentum tensor have completely determined the variables appearing in f_A . Therefore, the structures of all other moments of the distribution function are completely determined by the structures of the mass flux and the energy-momentum tensor. This is the hydrodynamical description of the gas as described by Israel and Stewart [16].

D. The Double Momentum and Entropy Fluxes

We shall now discuss the two other moments which are of physical interest. These are the double momentum flux and the entropy flux. Let us discuss the structure of the double momentum flux first.

We define the variables υ_A, ζ_A by

$$\upsilon_A \equiv - u_\alpha u_\beta u_\gamma U_A^{\alpha\beta\gamma} \; ; \quad \zeta_A \equiv - \Delta_{\alpha\beta} u_\gamma U_A^{\alpha\beta\gamma} \; . \tag{4.49}$$

As before, we use our knowledge of the structure of N_A and f_A to deduce the intermediate result

$$U_A^{\alpha\beta\gamma} = I_A^{\alpha\beta\gamma} + a_A J_A^{\alpha\beta\gamma} + \tilde{b}_A^{\lambda} J_{A\lambda}{}^{\alpha\beta\gamma} + \tilde{c}_A^{\lambda\tau} J_{A\lambda\tau}{}^{\alpha\beta\gamma} \; ; \tag{4.50}$$

which reduces after some algebra to

$$\begin{aligned} U_A^{\alpha\beta\gamma} = {} & \upsilon_A u^\alpha u^\beta u^\gamma + \frac{1}{3} \zeta_A \left(u^\alpha \Delta^{\beta\gamma} + u^\beta \Delta^{\alpha\gamma} + u^\gamma \Delta^{\alpha\beta}\right) \\ & + J_{A41}(u^\alpha u^\beta b_A^\gamma + u^\alpha u^\gamma b_A^\beta + u^\beta u^\gamma b_A^\alpha) \\ & + 2J_{A51}(u^\alpha u^\beta c_A^\gamma + u^\alpha u^\gamma c_A^\beta + u^\beta u^\gamma c_A^\alpha) \\ & + J_{A42}(\Delta^{\alpha\beta} b_A^\gamma + \Delta^{\alpha\gamma} b_A^\beta + \Delta^{\beta\gamma} b_A^\alpha) \\ & + 2J_{A52}(\Delta^{\alpha\beta} c_A^\gamma + \Delta^{\alpha\gamma} c_A^\beta + \Delta^{\beta\gamma} c_A^\alpha) \\ & + 2J_{A52}(u^\alpha c_A^{\beta\gamma} + u^\beta c_A^{\alpha\gamma} + u^\gamma c_A^{\alpha\beta}) \; ; \end{aligned} \tag{4.51}$$

where we have

$$\upsilon_A = I_{A30} + \delta\upsilon_A \; ; \; \delta\upsilon_A = J_{A30} a_A + J_{A40} b_A + (J_{A50} + J_{A51}) c_A \; ; \tag{4.52}$$

$$\begin{aligned} & \zeta_A = 3I_{A31} + \delta\zeta_A \; ; \\ & \delta\zeta_A = 3J_{A31} a_A + 3J_{A41} b_A + (3J_{A51} + 5J_{A52}) c_A \; . \end{aligned} \tag{4.53}$$

Since contraction over any two indices of $U_{A\alpha\beta\gamma}$ gives us $-M_A^\alpha$ we infer that

$$\upsilon_A = n_A + \zeta_A \;\; ; \;\; \delta\zeta_A = \delta n_A + \delta\zeta_A \;\; . \tag{4.54}$$

We may rewrite $\delta\zeta_A$ in terms of δn_A , $\delta\rho_A$, and π_A via equation (4.34). This gives us

$$\delta\zeta_A = U_{1A}\delta n_A + U_{2A}\delta\rho_A + U_{3A}\pi_A \;\; ; \tag{4.55}$$

where

$$U_{1A} \equiv 3J_{A31}\Omega_{A11} + 3J_{A41}\Omega_{A21} + (3J_{A51} + 5J_{A52})\Omega_{A31} \;\; ; \tag{4.56}$$

$$U_{2A} \equiv 3J_{A31}\Omega_{A12} + 3J_{A41}\Omega_{A22} + (3J_{A51} + 5J_{A52})\Omega_{A32} \;\; ; \tag{4.57}$$

$$U_{3A} \equiv 3J_{A31}\Omega_{A13} + 3J_{A41}\Omega_{A23} + (3J_{A51} + 5J_{A52})\Omega_{A33} \;\; . \tag{4.58}$$

Let us now use equation (4.34) to rewrite equation (4.51) while at the same time splitting it into first and zeroth order parts. We have, therefore, that

$$U_A^{\alpha\beta\gamma} = \overset{\circ}{U}{}_A^{\alpha\beta\gamma} + \delta U_A^{\alpha\beta\gamma} \;\; ; \tag{4.59}$$

$$\mathring{U}_A^{\alpha\beta\gamma} = (I_{A10}+3I_{A31})u^\alpha u^\beta u^\gamma + I_{A31}(u^\alpha\Delta^{\beta\gamma}+u^\beta\Delta^{\alpha\gamma}+u^\gamma\Delta^{\alpha\beta}) \;; \tag{4.60}$$

$$\begin{aligned}
\delta U_A^{\alpha\beta\gamma} = {} & (\delta\rho_A+\delta\zeta_A)u^\alpha u^\beta u^\gamma + \tfrac{1}{3}\delta\zeta_A(u^\alpha\Delta^{\beta\gamma} + u^\beta\Delta^{\alpha\gamma} + u^\gamma\Delta^{\alpha\beta}) \\
& + U_{4A}(u^\alpha u^\beta h_A^\gamma + u^\alpha u^\gamma h_A^\beta + u^\beta u^\gamma h_A^\alpha) \\
& + U_{5A}(u^\alpha u^\beta q_A^\gamma + u^\alpha u^\gamma q_A^\beta + u^\beta u^\gamma q_A^\alpha) \\
& + \frac{1}{5}(U_{4A}-\frac{1}{\eta_A})(\Delta^{\alpha\beta}h_A^\gamma + \Delta^{\alpha\gamma}h_A^\beta + \Delta^{\beta\gamma}h_A^\alpha) \\
& + \frac{1}{5}(U_{5A}-\frac{1}{\eta_A})(\Delta^{\alpha\beta}q_A^\gamma + \Delta^{\alpha\gamma}q_A^\beta + \Delta^{\beta\gamma}q_A^\alpha) \\
& + U_{6A}(\pi_A^{\alpha\beta}u^\gamma + \pi_A^{\alpha\gamma}u^\beta + \pi_A^{\beta\gamma}u^\alpha) \;;
\end{aligned} \tag{4.61}$$

where the coefficients U_{iA} $(i=4,5,6)$ are given by

$$U_{4A} \equiv J_{A41}\Omega_{A44} \;; \tag{4.62}$$

$$U_{5A} \equiv J_{A41}\Omega_{A45} + 2J_{A51}\Omega_{A55} \;; \tag{4.63}$$

$$U_{6A} \equiv 2J_{A52}\Omega_{A66} \;. \tag{4.64}$$

Equations (4.55) to (4.64) together comprise the structure of $U_A^{\alpha\beta\gamma}$.

The entropy flux structure is a little harder to deduce. The reason for this is that we shall need the entropy flux to second order to properly discuss the transient thermodynamics of the gas (Israel and Stewart [16]). This will be clarified in chapter V.

Now consider the function $\Phi_A(N_A)$ as given by equation (2.24) which we expand in a Taylor series to second order in δN_A:

$$\Phi_A(N_A) = \Phi_A(\mathring{N}_A) + \ln(\mathring{N}_A/\mathring{\Delta}_A)\delta N_A + \frac{1}{2}\frac{g_A}{\mathring{N}_A\mathring{\Delta}_A}(\delta N_A)^2 \quad . \tag{4.65}$$

Evaluating this expression further gives us

$$\begin{aligned}\Phi_A(N_A) = {} & (\alpha_A + \beta^\lambda_A w_{A\lambda})N_A + (\varepsilon_A^2 - 1)\mathring{N}_A \\ & - \varepsilon_A g_A \ln\{1-\varepsilon_A \exp(\alpha_A+\beta^\lambda_A w_{A\lambda})\} + \frac{1}{2}\frac{\mathring{N}_A\mathring{\Delta}_A}{g_A} f_A^2 \quad ;\end{aligned} \tag{4.66}$$

which reduces to the expression (3.13) in equilibrium. Inserting this expression into equation (2.22) gives us

$$S^\alpha_A = kI_{A21}\tilde{\beta}_A u^\alpha - k\alpha_A N^\alpha_A - k\tilde{\beta}^\lambda_A T_{A\lambda}{}^\alpha - Q^\alpha_A \quad ; \tag{4.67}$$

where we have defined the second order tensor Q^α_A by

$$Q^\alpha_A \equiv \frac{1}{2} k \int \frac{\mathring{N}_A\mathring{\Delta}_A}{g_A} f_A^2 \, w^\alpha_A \, dV_A \quad . \tag{4.68}$$

Introducing the expressions for the mass flux, the energy-momentum tensor, and the double-momentum tensor we obtain an alternative form for the entropy flux:

$$S^{\alpha}_{A} = S_{A}u^{\alpha} - \Theta_{A}j^{\alpha}_{A} + h^{\alpha}_{A}/T_{A} - Q^{\alpha}_{A} \quad ; \tag{4.69}$$

where

$$S_{A} = k\tilde{\beta}_{A}(I_{A21} + \rho_{A}) - \tilde{\Theta}_{A}n_{A} \quad . \tag{4.70}$$

The problem of deducing the structure of the entropy flux now reduces to deducing the structure of Q^{α}_{A}. We employ expression (4.2) to replace just one of the f_{A}'s appearing in the right-hand side of equation (4.68). We obtain, after some algebra, that

$$Q^{\alpha}_{A} = \frac{k}{2m_{A}}\left\{ a_{A}\delta M^{\alpha}_{A} + \tilde{b}^{\lambda}_{A}\delta T^{\alpha}_{A\lambda} + \tilde{c}^{\lambda\tau}_{A}\delta U_{A\lambda\tau}{}^{\alpha} \right\} \quad . \tag{4.71}$$

We re-express equation (4.71) via equations (4.20), (4.29), (4.61) and the definition (4.3):

$$\begin{aligned}
\frac{2}{k}m_{A}Q^{\alpha}_{A} = u^{\alpha}\Big\{ & a_{A}\delta n_{A} + c_{A}\delta n_{A} + b_{A}\delta\rho_{A} + \frac{4}{3}c_{A}\delta\zeta_{A} \\
& + (b^{\lambda}_{A} + 2U_{4A}c^{\lambda}_{A})h_{A\lambda} + 2U_{5A}c^{\lambda}_{A}q_{A\lambda} + U_{6A}c^{\lambda\tau}_{A}\pi_{A\lambda\tau}\Big\} \\
& + \Big\{ a_{A}/\eta_{A} + b_{A} + (c_{A}/3)(4U_{4A} - 1/\eta_{A}) \Big\} h^{\alpha}_{A} \\
& + (\pi_{A}/3)b^{\alpha}_{A} + \Big\{ -a_{A}/\eta_{A} + (c_{A}/3)(4U_{5A} + 1/\eta_{A}) \Big\} q^{\alpha}_{A} \\
& + \frac{2}{3}\delta\zeta_{A}c^{\alpha}_{A} + 2U_{6A}\pi^{\alpha\lambda}_{A}c_{A\lambda} + \frac{2}{5}(U_{4A} - 1/\eta_{A})c^{\alpha\lambda}_{A}h_{A\lambda} \\
& + \frac{2}{5}(U_{5A} + 1/\eta_{A})c^{\alpha\lambda}_{A}q_{A\lambda} + b_{A\lambda}\pi^{\lambda\alpha}_{A} \quad .
\end{aligned} \tag{4.72}$$

To help re-express equation (4.72) we note the following three relationships:

$$b^{\lambda}_{A} + 2U_{4A}c^{\lambda}_{A} = h^{\lambda}_{A}/J_{A31} \quad ; \tag{4.74}$$

$$a_{A}/\eta_{A} + b_{A} + (c_{A}/3)(4U_{4A} - 1/\eta_{A}) = \pi_{A}/(3J_{A31}) \quad ; \tag{4.75}$$

$$- a_{A}/\eta_{A} + (c_{A}/3)(4U_{4A} + 1/\eta_{A}) = \Omega_{A45}\pi_{A}/3 + \frac{2}{3}\Omega_{A55}\delta\zeta_{A} \quad . \tag{4.76}$$

We now use equations (4.34), (4.55) and the three relationships immediately above to re-express equation (4.72) and thereby obtain Q^{α}_{A} in its final form:

$$\begin{aligned} Q^{\alpha}_{A} = \frac{1}{2}\,u^{\alpha}\Big\{ & Q^{1}_{A}\pi^{2}_{A} + Q^{2}_{A}q^{\lambda}_{A}q_{A\lambda} + Q^{3}_{A}h^{\lambda}_{A}h_{A\lambda} + Q^{4}_{A}\pi^{\lambda\tau}_{A}\pi_{A\lambda\tau} \\ & + Q^{5}_{A}\pi_{A}\delta\rho_{A} + Q^{6}_{A}\delta\rho^{2}_{A} + Q^{7}_{A}\pi_{A}\delta n_{A} + Q^{8}_{A}\delta\rho_{A}\delta n_{A} + Q^{9}_{A}\delta n^{2}_{A}\Big\} \\ & + Q^{10}_{A}\pi_{A}q^{\alpha}_{A} + Q^{11}_{A}\pi_{A}h^{\alpha}_{A} + Q^{12}_{A}q^{\lambda}_{A}\pi_{A\lambda}{}^{\alpha} \\ & + Q^{13}_{A}h^{\lambda}_{A}\pi_{A\lambda}{}^{\alpha} + Q^{14}_{A}\delta\rho_{A}q^{\alpha}_{A} + Q^{15}_{A}\delta n_{A}q^{\alpha}_{A} \quad ; \end{aligned} \tag{4.77}$$

where the coefficients Q^{i}_{A} are given by

$$Q^{1}_{A} = \frac{k}{m_{A}}\frac{4}{3}U_{3A}\Omega_{A33} \quad ; \tag{4.78}$$

$$Q_A^2 = \frac{k}{m_A} 2U_{5A}\Omega_{A55} \quad ; \qquad (4.79)$$

$$Q_A^3 = \frac{k}{m_A} 1/J_{A31} \quad ; \qquad (4.80)$$

$$Q_A^4 = \frac{k}{m_A} U_{6A}\Omega_{A66} \quad ; \qquad (4.81)$$

$$Q_A^5 = \frac{k}{m_A}\left\{\Omega_{A23} + \frac{4}{3}\,\Omega_{A32}U_{3A} + \frac{4}{3}\,\Omega_{A33}U_{2A}\right\} \quad ; \qquad (4.82)$$

$$Q_A^6 = \frac{k}{m_A}\left(\Omega_{A22} + \frac{4}{3}\,\Omega_{A32}U_{2A}\right) \quad ; \qquad (4.83)$$

$$Q_A^7 = \frac{k}{m_A}\left\{\Omega_{A13} + \Omega_{A33} + \frac{4}{3}\,\Omega_{A31}U_{3A} + \frac{4}{3}\,\Omega_{A33}U_{1A}\right\} \quad ; \qquad (4.84)$$

$$Q_A^8 = \frac{k}{m_A}\left\{\Omega_{A12} + \Omega_{A31} + \frac{4}{3}\,\Omega_{A31}U_{2A} + \frac{4}{3}\,\Omega_{A32}U_{1A}\right\} \quad ; \qquad (4.85)$$

$$Q_A^9 = \frac{k}{m_A}\left\{\Omega_{A11} + \Omega_{A31} + \frac{4}{3}\,\Omega_{A31}U_{1A}\right\} \quad ; \qquad (4.86)$$

$$Q_A^{10} = \frac{k}{m_A}\left(\Omega_{A45}/3 + \frac{2}{3}\,U_{3A}\Omega_{A55}\right) \quad ; \qquad (4.87)$$

$$Q_A^{11} = \frac{k}{m_A} \frac{1}{3} \frac{1}{J_{A31}} \quad ; \tag{4.88}$$

$$Q_A^{12} = \frac{k}{m_A} \left\{ U_{6A} \Omega_{A55} + \frac{1}{5}(U_{5A} + 1/\eta_A) \Omega_{A66} + \frac{1}{2} \Omega_{A45} \right\} \quad ; \tag{4.89}$$

$$Q_A^{13} = \frac{k}{m_A} \left\{ \frac{1}{5}(U_{4A} - 1/\eta_A) \Omega_{A66} + \frac{1}{2} \Omega_{A44} \right\} = \frac{k}{m_A} \frac{1}{J_{A31}} \quad ; \tag{4.90}$$

$$Q_A^{14} = \frac{k}{m_A} \frac{2}{3} U_{2A} \Omega_{A55} \quad ; \tag{4.91}$$

$$Q_A^{15} = \frac{k}{m_A} \frac{2}{3} U_{1A} \Omega_{A55} \quad . \tag{4.92}$$

E. Invariance to First Order

We have deduced the non-equilibrium structures of the mass flux, the energy-momentum tensor, the double-momentum tensor, and the entropy flux. These four tensors describe all of the physics of our multi-component gas. The description of the physics must, however, be independent of how we separate these tensors into zeroth and first order parts, that is, the description must be invariant under the first order frame and fitting changes.

Let $\overset{*}{n}$ represent the number of species in our gas. Suppose we are given $2\overset{*}{n}$ sets of (measured) data (n_A, ρ_A) We select $2\overset{*}{n}$ numbers (α_A, β_A) and a rest frame u^α such that the fitting conditions specify the $2\overset{*}{n}$ numbers $(\delta n_A, \delta\rho_A)$ and our choice of u^α specifies (j^α_A, h^α_A).

The fitting changes and the frame changes tell us that our selection of (α_A, β_A) is not unique to first order. We could have specified another set (α'_A, β'_A) and another u'^α. Since these are different from our first set by order one, let us set

$$\alpha'_A = \alpha_A + \delta\alpha_A \; ; \; \beta'_A = \beta_A + \delta\beta_A \; ; \; u'^\alpha = u^\alpha + \delta u^\alpha \, . \qquad (4.93)$$

Then, invariance of $\ln(N_A/\Delta_A)$ requires that

$$f'_A = f_A - \delta f_A \; ; \; \delta f_A = \delta\alpha_A + (\delta\beta_A u^\lambda + \beta_A \delta u^\lambda) w_{A\lambda} \, . \qquad (4.94)$$

Let us put equations (4.93) into $\mathring{N}'_A$. We obtain $\mathring{N}'_A = \mathring{N}_A + \Delta N_A$ where $\Delta N_A = (\mathring{N}_A \mathring{\Delta}_A / g_A)\delta f_A$. Similarly, we can show that expression (4.12) becomes $\delta N'_A = \delta N_A - \Delta N_A$. Thus $N_A = \mathring{N}'_A + \delta N'_A = \mathring{N}_A + \delta N_A$ is invariant under fitting and frame changes. This decomposition of N_A is precisely the one used to obtain expressions for the mass flux, the energy-momentum tensor, and the double-momentum tensor. Hence, these tensors are invariant under fitting and frame changes.

The case of the entropy flux is more subtle because it is evaluated to second order. Equation (4.65) may be written in the primed system as

$$\Phi'_A(N_A) = \Phi'_{A0} + \Phi'_{A1} + \Phi'_{A2} \; ;$$
$$\Phi'_{A0} = \Phi_A(\mathring{N}'_A) \; ; \; \Phi'_{A1} = \ln(\mathring{N}'_A/\mathring{\Delta}'_A)\delta N'_A \; ; \; \Phi'_{A2} = \frac{1}{2}\frac{g_A}{\mathring{N}'_A\mathring{\Delta}'_A}\delta N'^{2}_A . \qquad (4.95)$$

Again, we expand the primed variables in terms of the unprimed variables via equation (4.93). Then we have that

$$\Phi'_{A0} = \Phi_{A0} + \ln(\mathring{N}_A/\mathring{\Delta}_A)\Delta N_A + \frac{1}{2}\frac{g_A}{\mathring{N}_A\mathring{\Delta}_A}\Delta N_A^2 \; ;$$
$$\Phi'_{A1} = \Phi_{A1} - \ln(\mathring{N}_A/\mathring{\Delta}_A)\Delta N_A + \frac{g_A}{\mathring{N}_A\mathring{\Delta}_A}\Delta N_A\delta N_A - \frac{g_A}{\mathring{N}_A\mathring{\Delta}_A}\Delta N_A^2 \; ; \qquad (4.96)$$
$$\Phi'_{A2} = \Phi_{A2} + \frac{g_A}{\mathring{N}_A\mathring{\Delta}_A}\Delta N_A\delta N_A + \frac{1}{2}\frac{g_A}{\mathring{N}_A\mathring{\Delta}_A}\Delta N_A^2 \; .$$

Addition of the last three equations above now tell us that

$$\Phi_A(N_A) = \Phi'_{A0} + \Phi'_{A1} + \Phi'_{A2} = \Phi_{A0} + \Phi_{A1} + \Phi_{A2} \; . \qquad (4.97)$$

Therefore the entropy flux is invariant to second order under fitting and frame changes.

We conclude that our description of the physics is independent of our choice of $(\alpha_A, \beta_A, u^\alpha)$ to first order. Consequently, we can choose any set of these quantities which is most convenient. However, we note that it is impossible to choose, in general, a frame such that all of the j^α_A's or the h^α_A's are zero.

There are two fitting choices that are most convenient. The first choice is $\delta n_A = \delta\rho_A = 0$ for all species. This means that I_{A10} and I_{A20} are the actual number density n_A and energy density ρ_A respectively. This has the advantage that $Q_A{}^{\alpha}$ is greatly simplified. However, we obtain a different temperature for each component. From this first fitting choice we may obtain the second choice by fitting changes to obtain a common $\tilde{\beta}$ for all species, which means that we will have a common temperature for all species. Let $(\mathring{\alpha}_A, \mathring{\beta}_A)$ be the values which imply that $\delta\mathring{n}_A = \delta\mathring{\rho}_A = 0$. Employing the first two equations in (4.93) tells us that

$$\begin{aligned} \delta n_A &= J_{A10}\delta\alpha_A - J_{A20}\delta\beta_A \ ; \\ \delta\rho_A &= J_{A20}\delta\alpha_A - J_{A30}\delta\beta_A \ . \end{aligned} \tag{4.98}$$

We wish to maintain $\delta n_A = 0$. This implies that $\delta\alpha_A = (J_{A20}/J_{A10})\delta\beta_A$ so that $\delta\rho_A = -(D_{A20}/J_{A10})\delta\beta_A$. We specify $\tilde{\beta}$ by requiring that

$$\sum_A \rho_A = \sum_A I_{A20}(\alpha_A, \beta_A) \leftrightarrow \sum_A \delta\rho_A = 0 \ . \tag{4.99}$$

Since $\delta\beta_A = m_A(\tilde{\beta} - \overset{\circ}{\tilde{\beta}}_A)$, equation (4.99) implies that

$$\tilde{\beta} = \left\{ \sum_A (D_{A20}\mathring{\beta}_A)/J_{A10} \right\} \div \left\{ \sum_A (D_{A20}m_A)/J_{A10} \right\} \ . \tag{4.100}$$

This completes our discussion of the non-equilibrium structures and the invariance of the physics. We shall now turn our attention to the discussion of the transient thermodynamics of the gas for massive species. This is performed in chapter V.

V. Transient Thermodynamics

So far, in our discussion of the gas, we have not considered the detailed structure of the covariant derivatives of the mass flux, the energy-momentum tensor, the double-momentum flux, or the entropy flux. A full discussion of those structures will require us to compute covariant derivatives of the first order quantities $(\delta n_A, \delta\rho_A, \pi_A, h^{\alpha}_A, q^{\alpha}_A, \pi^{\alpha\beta}_A)$. The retention of these derivatives in our calculations is a feature of transient thermodynamics as opposed to quasi-stationary theory.

Transient theory versus quasi-stationary theory may be schematically explained in terms of length scales [15,34]. Let λ represent the mean free path, the average distance a particle travels before it suffers a collision. Let L be the characteristic length over which macroscopic variables such as the number and energy densities change. We expect that, schematically, $1/L \sim p^{\mu}\nabla_{\mu}\mathring{N}/\mathring{N}$. Now let L′ be the characteristic distance over which first order quantities such as δn_A and $\delta\rho_A$ change appreciably. We expect that $1/L' \sim p^{\mu}\nabla_{\mu}f/f$. The relative importance of these scales may be deduced from a schematic analysis of the Boltzmann equation. For a single species gas of classical particles the collision term in the Boltzmann equation is $D_{coll}N \sim N^2\sigma f \sim Nf/\lambda$ where σ is the collision cross section. The Boltzmann equation is then schematically represented as $p^{\mu}\nabla_{\mu}N \sim Nf/\lambda$. With the substitution $N = \mathring{N}(1+f)$ for classical particles we have

$$p^{\mu}(\nabla_{\mu}\mathring{N}/\mathring{N}) + p^{\mu}(\nabla_{\mu}f/f)f + p^{\mu}(\nabla_{\mu}\mathring{N}/\mathring{N})f \sim f/\lambda \quad . \qquad (5.1)$$

In terms of our length scales this may be written as $1/L + f/L' + f/L \sim \frac{f}{\lambda}$ This shows that $f \sim \lambda/L << 1$; thus first order quantities are about the size of the ratio of the mean free path and the characteristic length scale of macroscopic quantities. The third term is of the size λ/L^2 which is much smaller than order one and negligible. Now if L′ is about the same size as L then the second term is also negligible; this is the realm of quasi-stationary theory. On the other hand, if L′ is about the size of the mean free path, then the second term is also of size 1/L and is not negligible; this is the realm of transient theory. In summary therefore, when first order quantities change over a scale which is large compared to the mean free path then they are negligible and we are in the realm of quasi-stationary theory. However, if they change over a scale which is comparable to the mean free path then they are not negligible and we are in the realm of transient theory.

A. Preliminary Discussion

Let us begin our discussion by defining some terminology which will be useful later on. If we let $\chi_{A}^{\alpha(n)}$ represent any tensor function where $\alpha(n) \equiv \alpha_1 \ldots \alpha_n$ then the time derivative and spatial derivative of $\chi_{A}^{\alpha(n)}$ are given by

$$\dot{\chi}_A^{\alpha(n)} \equiv \chi_A^{\alpha(n)}{}_{|\mu}\, u^{\mu} \quad ; \quad \nabla_{\mu}\chi_A^{\alpha(n)} \equiv \Delta_{\mu}{}^{\lambda}\chi_A^{\alpha(n)}{}_{|\lambda} \quad . \tag{5.2}$$

In particular, if $\chi_A^{\alpha(n)}$ is a scalar function of α_A and β_A, then the chain rule gives us

$$\chi_{A|\mu} = \left(\frac{\partial\chi_A}{\partial\alpha_A}\right)\alpha_{A|\mu} + \left(\frac{\partial\chi_A}{\partial\beta_A}\right)\beta_{A|\mu} \quad . \tag{5.3}$$

Now α_A and β_A are defined in non-equilibrium by the choice of the comparison equilibrium distribution function. In equilibrium we noted that $\alpha_{A|\mu} = 0$, $\beta_{A(\mu|\nu)} = 0$, and $\beta_{A|\mu} = \beta_A\dot{u}_\mu$. In a non-equilibrium state which is close to equilibrium we therefore expect that $\alpha_{A|\mu} = o_1$, $\beta_{A(\mu|\nu)} = o_1$, and $\beta_{A|\mu} = \beta_A\dot{u}_\mu + o_1$. Hence equation (5.3) may be rewritten as

$$\chi_{A|\mu} = \bar{\chi}_A\dot{u}_\mu + o_1 \quad ; \tag{5.4}$$

where we have defined

$$\bar{\chi}_A \equiv \beta_A\left(\frac{\partial\chi_A}{\partial\beta_A}\right) . \tag{5.5}$$

The time derivative of χ_A may now be seen to be of order one: $\dot{\chi}_A(\alpha_A, \beta_A) = o_1$. This property, along with definition (5.5) and equation (5.4) will simplify the computation of the derivative of the double-momentum tensor.

Let us also define the quantity $\tilde{\eta}_A$ by

$$\tilde{\eta}_A \equiv \beta_A \frac{\partial}{\partial \beta_A}(1/\eta_A) \quad . \tag{5.6}$$

Let us also note a number of useful relationships:

$$\dot{h}^{\alpha}_{A} = \dot{h}^{\lambda}_{A} u_{\lambda} u^{\alpha} + \Delta^{\alpha}{}_{\lambda} \dot{h}^{\lambda}_{A} \; ; \; h^{\alpha}_{A} \dot{u}_{\alpha} = - \dot{h}^{\alpha}_{A} u_{\alpha} \; ; \tag{5.7}$$

$$\dot{q}^{\alpha}_{A} = \dot{q}^{\lambda}_{A} u_{\lambda} u^{\alpha} + \Delta^{\alpha}{}_{\lambda} \dot{q}^{\lambda}_{A} \; ; \; q^{\alpha}_{A} \dot{u}_{\alpha} = - \dot{q}^{\alpha}_{A} u_{\alpha} \; ; \tag{5.8}$$

$$u_{\alpha} \pi^{\alpha\lambda}_{A}{}_{|\lambda} = 0_2 \; ; \; \Delta_{\mu\alpha} \pi^{\alpha\lambda}_{A}{}_{|\lambda} = \pi_{A\mu}{}^{\lambda}{}_{|\lambda} + 0_2 \; ; \tag{5.9}$$

$$\Delta_{\mu\alpha} u_{\beta} \nabla^{\alpha} h^{\beta}_{A} = - h^{\lambda}_{A} \omega_{\lambda\mu} + 0_2 \; ; \; \Delta_{\mu\alpha} u_{\beta} \nabla^{\alpha} q^{\beta}_{A} = - q^{\lambda}_{A} \omega_{\lambda\mu} + 0_2 \; ; \tag{5.10}$$

$$u_{\alpha} \dot{\pi}^{\alpha\lambda}_{A} = - \dot{u}_{\alpha} \pi^{\alpha\lambda}_{A} + 0_2 \; ; \; \Delta_{\mu\alpha} \dot{\pi}^{\alpha\lambda}_{A} = - u_{\mu} \dot{u}_{\alpha} \pi^{\alpha\lambda}_{A} + \dot{\pi}_{A\mu}{}^{\lambda} + 0_2 \; . \tag{5.11}$$

We define the symmetric, spatial, trace free part of an arbitrary second rank tensor $X^{\alpha\beta}_{A}$ by

$$X^{<\alpha\beta>}_{A} \equiv (\Delta^{\alpha}{}_{\lambda} \Delta^{\beta}{}_{\tau} + \Delta^{\alpha}{}_{\tau} \Delta^{\beta}{}_{\lambda} - \frac{2}{3} \Delta_{\lambda\tau} \Delta^{\alpha\beta}) X^{\lambda\tau}_{A} \quad . \tag{5.12}$$

Then, for example, $u_{<\alpha|\beta>} = 2\sigma_{\alpha\beta}$. We have completed our preliminary discussion and shall now consider the major topics of this chapter.

B. The Derivatives of the Physical Tensors

The first and zeroth order parts of the mass flux as given by equations (4.19) and (4.20) may be differentiated and contracted to give us the following two equations, correct to order one:

$$\overset{\circ}{M}^{\alpha}_{A|\alpha} = \dot{I}_{A10} + \hat{I}_{A10}\theta \quad ; \tag{5.13}$$

$$\delta M^{\alpha}_{A|\alpha} = \delta\dot{n}_{A} + j^{\alpha}_{A|\alpha} \quad . \tag{5.14}$$

We may solve for j^{α}_{A} in terms of h^{α}_{A} and q^{α}_{A} via equation (4.30). Differentiation and contraction of that solution now informs us that

$$j^{\alpha}_{A|\alpha} = (1/\eta_{A})h^{\alpha}_{A|\alpha} - (1/\eta_{A})q^{\alpha}_{A|\alpha} + \tilde{\eta}_{A}(h^{\alpha}_{A} - q^{\alpha}_{A})\dot{u}_{\alpha} \quad ; \tag{5.15}$$

where we have employed definition (5.6) and equation (5.4) to produce the last term. Substituting equation (5.15) into equation (5.14) produces

$$\delta M^{\alpha}_{A|\alpha} = \delta\dot{n}_{A} + (1/\eta_{A})h^{\alpha}_{A|\alpha} - (1/\eta_{A})q^{\alpha}_{A|\alpha} + \bar{\eta}_{A}(h^{\alpha}_{A}-q^{\alpha}_{A})\dot{u}_{\alpha} \, . \tag{5.16}$$

The zeroth and first order parts of the energy-momentum tensor as given by equations (4.28) and (4.29) may be differentiated and contracted to produce expressions for $\mathring{T}^{\alpha\lambda}_{A}|_{\lambda}$ and $\delta T^{\alpha\lambda}_{A}|_{\lambda}$. These two expressions are now contracted with u^{α} and $\Delta^{\mu\alpha}$ to produce four equations, correct to first order:

$$- u_{\alpha}\mathring{T}^{\alpha\lambda}_{A}|_{\lambda} = \dot{I}_{A20} + \eta_{A}I_{A10}\theta \quad ; \qquad (5.17)$$

$$\Delta_{\mu\alpha}\mathring{T}^{\alpha\lambda}_{A}|_{\lambda} = \nabla_{\mu}I_{A21} + \eta_{A}I_{A10}\dot{u}_{\mu} \quad ; \qquad (5.18)$$

$$- u_{\alpha}\delta T^{\alpha\lambda}_{A}|_{\lambda} = \delta\dot{\rho}_{A} + h^{\lambda}_{A|\lambda} + h^{\alpha}_{A}\dot{u}_{\alpha} \quad ; \qquad (5.19)$$

$$\begin{aligned} \Delta_{\mu\alpha}\delta T^{\alpha\lambda}_{A}|_{\lambda} = {} & \frac{1}{3}\nabla_{\mu}\pi_{A} + (\delta\rho_{A} + \pi_{A}/3)\dot{u}_{\mu} \\ & + \Delta_{\mu}{}^{\lambda}\dot{h}_{A\lambda} - h^{\lambda}_{A}\omega_{\lambda\mu} + \pi_{A\mu}{}^{\lambda}|_{\lambda} \quad ; \end{aligned} \qquad (5.20)$$

where we have used relationships (5.7) and (5.9).

The case of the double-momentum tensor is treated similarly to the energy-momentum case above. We differentiate and contract equations (4.60) and (4.61) to obtain expressions for $\mathring{U}^{\alpha\beta\gamma}_{A}|_{\gamma}$ and $\delta U^{\alpha\beta\gamma}_{A}|_{\gamma}$. These two expressions are contracted with $u_{\alpha}u_{\beta}$, $u_{\alpha}\Delta_{\mu\beta}$, and $\Delta_{\mu\alpha}\Delta_{\nu\beta} - \frac{1}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta}$; this gives us six equations:

$$u_{\alpha}u_{\beta}\mathring{U}^{\alpha\beta\gamma}_{A}|_{\gamma} = (\dot{I}_{A10} + I_{A10}\theta) + (3\dot{I}_{A31} + 5I_{A31}\theta) \quad ; \qquad (5.21)$$

$$- \Delta_{\mu\alpha} u_\beta \mathring{U}_A^{\alpha\beta\gamma}\big|_\gamma = 3\nabla_\mu I_{A31} + (I_{A10} + 5I_{A31})\dot{u}_\mu \quad ; \tag{5.22}$$

$$(\Delta_{\mu\alpha}\Delta_{\nu\beta} - \tfrac{1}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta})\mathring{U}_A^{\alpha\beta\gamma}\big|_\gamma = I_{A31} u_{<\mu|\nu>} \quad ; \tag{5.23}$$

$$\begin{aligned} u_\alpha u_\beta \delta U_A^{\alpha\beta\gamma}\big|_\gamma = {} & \delta\dot{n}_A + \delta\dot{\zeta}_A + (2U_{4A} + \bar{U}_{4A})h_A^\lambda \dot{u}_\lambda \\ & + (2U_{5A} + \bar{U}_{5A})q_A^\lambda \dot{u}_\lambda + U_{4A} h_A^\lambda\big|_\lambda + U_{5A} q_A^\lambda\big|_\lambda \quad ; \end{aligned} \tag{5.24}$$

$$\begin{aligned} - \Delta_{\mu\alpha} u_\beta \delta U_A^{\alpha\beta\gamma}\big|_\gamma = {} & \frac{1}{3}\nabla_\mu \delta\zeta_A + \frac{5}{3}\delta\zeta_A \dot{u}_\mu + U_{4A}\Delta_{\mu\lambda}\dot{h}_A^\lambda \\ & + U_{5A}\Delta_{\mu\lambda}\dot{q}_A^\lambda - U_{4A} h_A^\lambda \omega_{\lambda\mu} - U_{5A} q_A^\lambda \omega_{\lambda\mu} \\ & + U_{6A}\pi_{A\mu}{}^\lambda\big|_\lambda + (U_{6A} + \bar{U}_{6A})\pi_{A\mu}{}^\lambda \dot{u}_\lambda \quad ; \end{aligned} \tag{5.25}$$

$$\begin{aligned} (\Delta_{\mu\alpha}\Delta_{\nu\beta} - \frac{1}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta})\delta U^{\alpha\beta\gamma}\big|_\gamma = I_{A31} \Big\{ & t_{A30}\nabla_{<\mu} h_{A\nu>} \\ & + t_{A31}\nabla_{<\mu} q_{A\nu>} + t_{A32} h_{A<\mu}\dot{u}_{\nu>} \\ & + t_{A33} q_{A<\mu}\dot{u}_{\nu>} + t_{A34}\dot{\pi}_{A<\mu\nu>} - 2t_{A34}\pi_{A<\mu}{}^\lambda \omega_{\lambda\nu>} \quad ; \end{aligned} \tag{5.26}$$

where we have used relationships (5.9), (5.10), (5.11) and definition (5.12) as well as defining the following coefficients:

$$t_{A30} \equiv \frac{1}{I_{A31}}\frac{1}{5}(U_{4A} - 1/\eta_A) \quad ; \tag{5.27}$$

$$t_{A31} \equiv \frac{1}{I_{A31}}\frac{1}{5}(U_{5A} + 1/\eta_A) \quad ; \tag{5.28}$$

$$t_{A32} \equiv \frac{1}{I_{A31}} \left\{ \frac{1}{5}(U_{4A} - 1/\eta_A) + U_{4A} + \frac{1}{5}(\bar{U}_{4A} - \bar{\eta}_A) \right\} \quad ; \tag{5.29}$$

$$t_{A33} \equiv \frac{1}{I_{A31}} \left\{ \frac{1}{5}(U_{5A} + 1/\eta_A) + U_{5A} + \frac{1}{5}(\bar{U}_{5A} + \bar{\eta}_A) \right\} \quad ; \tag{5.30}$$

$$t_{A34} \equiv \frac{1}{2} U_{6A}/I_{A31} \quad . \tag{5.31}$$

In equations (5.24) and (5.25) we have retained $\delta\dot{\zeta}_A$ and $\nabla_\mu \delta\zeta_A$ for the sake of simplicity. We can obtain expressions for these quantities by differentiating equation (4.55). We then have, correct to first order, the following two results:

$$\delta\dot{\zeta}_A = U_{1A}\delta\dot{n}_A + U_{2A}\delta\dot{\rho}_A + U_{3A}\dot{\pi}_A \quad ; \tag{5.32}$$

$$\begin{aligned} \nabla_\mu \delta\zeta_A = {} & U_{1A}\nabla_\mu \delta n_A + U_{2A}\nabla_\mu \delta\rho_A + U_{3A}\nabla_\mu \pi_A \\ & + \left\{ \bar{U}_{1A}\delta n_A + \bar{U}_{2A}\delta\rho_A + \bar{U}_{3A}\pi_A \right\} \dot{u}_\mu \quad ; \end{aligned} \tag{5.33}$$

where we have taken into account that $\dot{U}_{1A}$ is order one and we have used equation (5.4).

C. The Entropy Production

We now have sufficient information from the above calculations to intelligently discuss the entropy production, $S^\alpha_{A|\alpha}$, for species A. As a first step, we shall consider $Q^\alpha_{A|\alpha}$. We shall require all computations of $Q^\alpha_{A|\alpha}$ to be correct to second order.

To compute $Q^{\alpha}_{A|\alpha}$ we consider equation (4.77). Straightforward differentiation and contraction and using definition (5.2) provides us with the following result:

$$\begin{aligned} Q^{\alpha}_{A|\alpha} = {} & \overset{1}{Q}_{A}\pi_{A}\dot{\pi}_{A} + \overset{2}{Q}_{A}q^{\alpha}_{A}\dot{q}_{A\alpha} + \overset{3}{Q}_{A}h^{\alpha}_{A}\dot{h}_{A\alpha} + \overset{4}{Q}_{A}\pi^{\alpha\beta}_{A}\dot{\pi}_{A\alpha\beta} \\ & + \frac{1}{2}\,\overset{5}{Q}_{A}\dot{\pi}_{A}\delta\rho_{A} + \frac{1}{2}\,\overset{5}{Q}_{A}\pi_{A}\delta\dot{\rho}_{A} + \overset{6}{Q}_{A}\delta\rho_{A}\delta\dot{\rho}_{A} + \frac{1}{2}\,\overset{7}{Q}_{A}\dot{\pi}_{A}\delta n_{A} \\ & + \frac{1}{2}\,\overset{7}{Q}_{A}\pi_{A}\delta\dot{n}_{A} + \frac{1}{2}\,\overset{8}{Q}_{A}\delta\dot{\rho}_{A}\delta n_{A} + \frac{1}{2}\,\overset{8}{Q}_{A}\delta\rho_{A}\delta\dot{n}_{A} + \overset{9}{Q}_{A}\delta n_{A}\delta\dot{n}_{A} \\ & + Q^{10}_{A}\,\pi_{A|\alpha}q^{\alpha}_{A} + Q^{10}_{A}\,\pi_{A}q^{\alpha}_{A|\alpha} + \bar{Q}^{10}_{A}\,\pi_{A}q^{\alpha}_{A}\dot{u}_{\alpha} \\ & + Q^{11}_{A}\,\pi_{A|\alpha}h^{\alpha}_{A} + Q^{11}_{A}\,\pi_{A}h^{\alpha}_{A|\alpha} + \bar{Q}^{11}_{A}\,\pi_{A}h^{\alpha}_{A}\dot{u}_{\alpha} \\ & + Q^{12}_{A}\,\pi^{\alpha\lambda}_{A}{}_{|\lambda}q_{A\alpha} + Q^{12}_{A}\,\pi^{\alpha\lambda}_{A}q_{A\alpha|\lambda} + \bar{Q}^{12}_{A}\,\pi^{\alpha\lambda}_{A}q_{A\alpha}\dot{u}_{\lambda} \\ & + Q^{13}_{A}\,\pi^{\alpha\lambda}_{A}{}_{|\lambda}h_{A\alpha} + Q^{13}_{A}\,\pi^{\alpha\lambda}_{A}h_{A\alpha|\lambda} + \bar{Q}^{13}_{A}\,\pi^{\alpha\lambda}_{A}q_{A\alpha}\dot{u}_{\lambda} \\ & + Q^{14}_{A}\,q^{\alpha}_{A}\delta\rho_{A|\alpha} + Q^{14}_{A}\,\delta\rho_{A}q^{\alpha}_{A|\alpha} + \bar{Q}^{14}_{A}\,\delta\rho_{A}q^{\alpha}_{A}\dot{u}_{\alpha} \\ & + Q^{15}_{A}\,q^{\alpha}_{A}\delta n_{A|\alpha} + Q^{15}_{A}\,\delta n_{A}q^{\alpha}_{A|\alpha} + \bar{Q}^{15}_{A}\,\delta n_{A}q^{\alpha}_{A}\dot{u}_{\alpha}\ . \end{aligned} \tag{5.34}$$

We can, however, compute $Q^{\alpha}_{A|\alpha}$ by another method. If we differentiate the original definition of Q^{α}_{A}, equation (4.68), we obtain the following expression:

$$Q^{\alpha}_{A|\alpha} = \frac{k}{2}\int \left\{(\overset{\circ}{N}_{A}\overset{\circ}{\Delta}_{A}/g_{a})f^{2}_{A}\right\}_{||\alpha}w^{\alpha}_{A}\,dV_{A} \quad ; \tag{5.35}$$

where we have brought the covariant derivative inside the integral. The integrand may be expressed as

$$(\overset{\circ}{N}_{A}\overset{\circ}{\Delta}_{A}f^{2}_{A}/g_{a})_{||\alpha}w^{\alpha}_{A} = 2f_{A}\left\{\frac{\overset{\circ}{N}_{A}\overset{\circ}{\Delta}_{A}}{g_{A}}f_{A}\right\}_{||\alpha}w^{\alpha}_{A} - \left\{\frac{\overset{\circ}{N}_{A}\overset{\circ}{\Delta}_{A}}{g_{A}}\right\}_{||\alpha}w^{\alpha}_{A}f^{2}_{A}\ . \tag{5.36}$$

The second term on the right-hand side of this last equation may be rewritten as

$$f_A^2\left\{\frac{\mathring{N}_A\mathring{\Delta}_A}{g_A}\right\}\Big|\Big|_\alpha w_A^\alpha = f_A^2/g_A\,\frac{d(\mathring{N}_A\mathring{\Delta}_A)}{d\{\ln(N_A/\Delta_A)\}}\{\ln(\mathring{N}_A/\mathring{\Delta}_A)\}\Big|\Big|_\alpha w_A^\alpha$$

$$= \frac{f_A^2}{g_A}\,\frac{d(\mathring{N}_A\mathring{\Delta}_A)}{d\{\ln(\mathring{N}_A/\mathring{\Delta}_A)}\,(\alpha_{A|\mu} + 2\beta_{A(\mu|\nu)}w_A^\nu)w_A^\mu \; . \tag{5.37}$$

Since $\alpha_{A|\mu}$ and $\beta_{A(\mu|\nu)}$ are both first order quantities, we see that the quantity expressed by equation (5.37) is of order three and therefore negligible. Therefore, from the the three equations immediately above, we conclude that

$$Q^\alpha_{A|\alpha} = k\int f_A(\mathring{N}_A\mathring{\Delta}_A f_A/g_A)\Big|\Big|_\alpha w_A^\alpha \, dV_A \; . \tag{5.38}$$

Let us now re-express the undifferentiated f_A in equation (5.38) via equation (4.2); this allows us to bring the covariant derivative outside the integral. We eventually obtain the following result:

$$Q^\alpha_{A|\alpha} = \frac{k}{m_A}\left\{ a_A\delta M^\alpha_{A|\alpha} + \tilde{b}^\lambda_A\delta T_{A\lambda}{}^\alpha{}_{|\alpha} + \tilde{c}^{\lambda\tau}_A\delta U_{A\lambda\tau}{}^\alpha{}_{|\alpha} \right\} \; . \tag{5.39}$$

We may rearrange this last equation in terms of quantities that we have already derived:

$$
\begin{aligned}
Q^{\alpha}_{A|\alpha} = \frac{k}{m_A} \Big\{ & (a_A - c_A/3)\, \delta M^{\alpha}_{A|\alpha} + b_A \, (-u_\lambda \delta T^{\lambda\alpha}_{A}{}_{|\alpha}) \\
& + \frac{4}{3} c_A \, (u_\alpha u_\beta \delta U^{\alpha\beta\gamma}_{A}{}_{|\alpha}) + b^{\lambda}_{A} \, (\Delta_{\lambda\alpha} \delta T^{\alpha\beta}_{A}{}_{|\beta}) \\
& + 2\, c^{\lambda}_{A} \, (- \Delta_{\lambda\alpha} u_\beta \delta U^{\alpha\beta\gamma}_{A}{}_{|\gamma}) \\
& + c^{\lambda\tau}_{A} \, \{(\Delta_{\lambda\alpha}\Delta_{\tau\beta} - \frac{1}{3} \Delta_{\lambda\tau}\Delta_{\alpha\beta}) \delta U^{\alpha\beta\gamma}_{A}{}_{|\gamma}\} \quad .
\end{aligned}
\tag{5.40}
$$

We now substitute into equation (5.40) equations (5.16), (5.19), (5.20), (5.24), (5.25), and (5.26); we also replace the remaining variables via equation (4.34). The resulting expression, which is too complicated to reproduce here, must agree with the right hand side of equation (5.34). A detailed comparison provides us with a small number of new relationships:

$$
\begin{aligned}
\frac{1}{2} Q^{5}_{A} &= \frac{k}{m_A} \frac{4}{3} \Omega_{A32} U_{3A} \\
&= \frac{k}{m_A} (\Omega_{A23} + \frac{4}{3} \Omega_{A33} U_{2A}) \quad ;
\end{aligned}
\tag{5.41}
$$

$$
\begin{aligned}
\frac{1}{2} Q^{7}_{A} &= \frac{k}{m_A} \frac{4}{3} \Omega_{A31} U_{3A} \\
&= \frac{k}{m_A} (\Omega_{A13} + \Omega_{A33} + \frac{4}{3} \Omega_{A33} U_{1A}) \quad ;
\end{aligned}
\tag{5.42}
$$

$$
\begin{aligned}
\frac{1}{2} Q^{8}_{A} &= \frac{k}{m_A} (\Omega_{A21} + \frac{4}{3} \Omega_{A31} U_{2A}) \\
&= \frac{k}{m_A} (\Omega_{A12} + \Omega_{A32} + \frac{4}{3} \Omega_{A32} U_{1A}) \quad ;
\end{aligned}
\tag{5.43}
$$

$$\bar{Q}_A^{10} = \frac{k}{m_A}\left\{ \tilde{\eta}_A \Omega_{A13} + \frac{\Omega_{A33}}{3}[4(2U_{5A} + \bar{U}_{5A}) + \tilde{\eta}_A] + \Omega_{A45} - 2\Omega_{A55}\left(\bar{U}_{3A} + \frac{5}{3}U_{3A}\right) \right\} ; \tag{5.44}$$

$$\bar{Q}_A^{11} = \frac{k}{m_A}\left\{ -\tilde{\eta}_A \Omega_{A13} + \frac{\Omega_{A33}}{3}[4(2U_{4A} + \bar{U}_{4A}) - \tilde{\eta}_A] + \frac{\Omega_{A44}}{3} \right\} ; \tag{5.45}$$

$$\bar{Q}_A^{12} = \frac{k}{m_A}\left\{ 2I_{A31}t_{A30}\Omega_{A66} - 2\Omega_{A55}(U_{6A} + \bar{U}_{6A}) \right\} ; \tag{5.46}$$

$$\bar{Q}_A^{13} = \frac{k}{m_A} 2I_{A31}t_{A31}\Omega_{A66} ; \tag{5.47}$$

$$\bar{Q}_A^{14} = \frac{k}{m_A}\left\{ \tilde{\eta}_A \Omega_{A12} + \frac{\Omega_{A32}}{3}[4(2U_{5A} + \bar{U}_{5A}) + \tilde{\eta}_A] + \Omega_{A45} - 2\Omega_{A55}\left(\bar{U}_{2A} + \frac{5}{3}U_{2A}\right) \right\} ; \tag{5.48}$$

$$\bar{Q}_A^{15} = \frac{k}{m_A}\left\{ \tilde{\eta}_A \Omega_{A11} + \frac{\Omega_{A31}}{3}[4(2U_{5A} + \bar{U}_{5A}) + \tilde{\eta}_A] - 2\Omega_{A55}\left(\bar{U}_{1A} + \frac{5}{3}U_{2A}\right) \right\} . \tag{5.49}$$

At this point, we should note that even though Q_A^α is second order, its derivative produces second order quantities. Hence, we needed to know the entropy flux to second order to correctly compute the entropy production to second order.

Let us now compute the entropy production for species A. Equation (4.67) may be differentiated to give us

$$S^\alpha_{A|\alpha} = \left\{ \frac{k}{m_A} I_{A21}\beta_A u^\alpha - \Theta_A M^\alpha_A - \frac{k}{m_A}\beta_A u^\lambda T_{A\lambda}{}^\alpha \right\}_{|\alpha} - Q^\alpha_{A|\alpha} . \tag{5.50}$$

Performing the differentiation on the first term in this equation eventually leads to the following result:

$$\begin{aligned} S^{\alpha}_{A|\alpha} = &- \frac{k}{m_A} \beta_A \frac{\pi_A}{3} \theta + \frac{1}{\eta_A} q^{\alpha}_A \nabla_{\alpha} \Theta_A - \frac{k}{m_A} \beta_A \sigma_{\alpha\beta} \pi^{\alpha\beta}_A \\ &- \frac{k}{m_A} \beta_A u^{\lambda} T_{A\lambda}{}^{\alpha}{}_{|\alpha} - \frac{k}{m_A} \delta n_A \dot{\alpha}_A + \frac{k}{m_A} \delta \rho_A \dot{\beta}_A \\ &- \frac{k}{m_A} \frac{1}{J_{A31}} h^{\alpha}_A (\nabla_{\alpha} I_{A21} + \eta_A I_{A10} \dot{u}_{\alpha}) - Q^{\alpha}_{A|\alpha} . \end{aligned} \tag{5.51}$$

To check this expression, we compute the entropy production by another method. From the definition of the entropy flux we have

$$S^{\alpha}_{A|\alpha} = \int \{\Phi_A(N_A)\}_{||\alpha} w^{\alpha}_A \, dV_A \quad ; \tag{5.52}$$

where $\Phi_A(N_A)$ is given by equation (2.24). Since we have that

$$\{\Phi_A(N_A)\}_{||\alpha} = \{\ln(N_A/\Delta_A)\} N_{A||\alpha} \quad , \tag{5.53}$$

then equation (5.52) may be written, via equations (4.1) and (4.2), as

$$\begin{aligned} S^{\alpha}_{A|\alpha} = &-k(\alpha_A + \frac{a_A}{m_A}) M^{\alpha}_{A|\alpha} - k(\tilde{\beta}^{\lambda}_A + \frac{\tilde{b}^{\lambda}_A}{m_A}) T_{A\lambda}{}^{\alpha}{}_{|\alpha} \\ &- \frac{k}{m_A} \tilde{c}^{\lambda\tau}_A U_{A\lambda\tau}{}^{\alpha}{}_{|\alpha} . \end{aligned} \tag{5.54}$$

We can separate out $Q^{\alpha}_{A|\alpha}$ in equation (5.54) by virtue of equation (5.39):

$$
\begin{aligned}
S^{\alpha}_{A|\alpha} = {} & -k(\alpha_A + a_A/m_A)\overset{\circ}{M}{}^{\alpha}_{A|\alpha} + \frac{k}{m_A}\,\beta_A(-u^{\lambda}\overset{\circ}{T}{}_{A\lambda}{}^{\alpha}|_{\alpha} - u^{\lambda}\delta T_{A\lambda}{}^{\alpha}|_{\alpha}) \\
& - \frac{k}{m_A}\,b_A(-u^{\lambda}\overset{\circ}{T}{}_{A\lambda}{}^{\alpha}|_{\alpha}) + \frac{k}{m_A}\,b^{\lambda}_A(-\Delta_{\lambda\alpha}\overset{\circ}{T}{}^{\alpha\beta}_A|_{\beta}) \\
& - \frac{4}{3}\,c_A(u_\alpha u_\beta \overset{\circ}{U}{}^{\alpha\beta\gamma}_A|_{\gamma}) - \frac{2k}{m_A}\,c^{\lambda}_A(-\Delta_{\lambda\alpha}u_\beta\overset{\circ}{U}{}^{\alpha\beta\gamma}_A|_{\gamma}) \\
& - \frac{k}{m_A}c^{\lambda\tau}_A[(\Delta_{\lambda\alpha}\Delta_{\tau\beta} - \frac{1}{3}\,\Delta_{\lambda\tau}\Delta_{\alpha\beta})\overset{\circ}{U}{}^{\alpha\beta\gamma}_A|_{\gamma}] - Q^{\alpha}_{A|\alpha} \quad .
\end{aligned}
\qquad (5.55)
$$

We may now substitute into equation (5.55) equations (5.13), (5.17), (5.18), (5.19), (5.21), (5.22), and (5.23) as well as employing equation (4.34) to replace the remaining variables. This whole procedure produces equation (5.51). Hence we confirm that equation (5.51) is the correct expression for the entropy production.

Equation (5.51) may be re-expressed for better clarity. We add equations (5.18) and (5.20) and rearrange the result to obtain the following expression:

$$
\begin{aligned}
\nabla_\mu I_{A21} + \eta_A I_{A10}\dot{u}_\mu = {} & \Delta_{\mu\alpha}T^{\alpha\lambda}_A|_{\lambda} - \Big\{ \frac{1}{3}\,\nabla_\mu \pi_A \\
& + (\delta\rho_A + \frac{\pi_A}{3})\dot{u}_\mu + \Delta_{\mu\lambda}\dot{h}^{\lambda}_A - h^{\lambda}_A\omega_{\lambda\mu} + \pi_{A\mu}{}^{\lambda}|_{\lambda}\Big\} \quad .
\end{aligned}
\qquad (5.56)
$$

We now substitute this expression into equation (5.51) to obtain the following result:

$$
\begin{aligned}
S^{\alpha}_{A|\alpha} = {} & -\frac{k}{m_A}\,\beta_A\pi_A\theta/3 + \frac{1}{\eta_A}\,q^{\lambda}_A\nabla_\lambda\theta_A - \frac{k}{m_A}\,\beta_A\sigma_{\alpha\beta}\pi^{\alpha\beta}_A - \frac{1}{J_{A31}}\,h^{\lambda}_A\nabla_\lambda\pi_A \\
& -(\delta\rho_A + \pi_A/3)\frac{1}{J_{A31}}\,h^{\lambda}_A\dot{u}_\lambda - \frac{1}{J_{A31}}\,h^{\lambda}_A\dot{h}_{A\lambda} - \frac{k}{m_A}\,\delta n_A\dot{\alpha}_A \\
& + \frac{k}{m_A}\,\delta\rho_A\dot{\beta}_A + \frac{1}{J_{A31}}\,h^{\lambda}_A T_{A\lambda}{}^{\alpha}|_{\alpha} - \frac{k}{m_A}\,\beta^{\lambda}_A T_{A\lambda}{}^{\alpha}|_{\alpha} - Q^{\alpha}_{A|\alpha} \quad .
\end{aligned}
\qquad (5.57)
$$

In the case of the single species gas, $T^{\alpha\lambda}_{A}{}_{|\lambda} = 0$, equation (5.57) reduces to the result as given by Israel and Stewart [16]. Also, when we neglect the transient terms, we obtain precisely the stationary result in kinetic theory given by Israel [12].

D. Transport Equation Preliminaries

Let us now consider the transport equations which are obtained from the three components of $U^{\alpha\beta\lambda}_{A}{}_{|\lambda}$, that is $u_\alpha u_\beta U^{\alpha\beta\lambda}_{A}{}_{|\lambda}$, $\Delta_{\mu\alpha} u_\beta U^{\alpha\beta\lambda}_{A}{}_{|\lambda}$, and $(\Delta_{\mu\alpha}\Delta_{\nu\beta} - \frac{1}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta}) U^{\alpha\beta\lambda}_{A}{}_{|\lambda}$. Before we can derive these, we must re-express equations (5.21) and (5.22) in appropriate forms.

We begin by adding equations (5.13) and (5.16) and rearranging the result to produce the following expression:

$$\begin{aligned} \dot{I}_{A10} = &- I_{A10}\theta - \delta\dot{n}_A - (1/\eta_A) h^{\alpha}_{A|\alpha} \\ &+ (1/\eta_A) q^{\alpha}_{A|\alpha} - \bar{\eta}_A (h^{\alpha}_{A} - q^{\alpha}_{A})\dot{u}_\alpha \ . \end{aligned} \tag{5.58}$$

Similarily, we add equations (5.17) and (5.19) to obtain the following result:

$$\dot{I}_{A20} = - \eta_A I_{A10}\theta - \delta\dot{\rho}_A - h^{\alpha}_{A|\alpha} - h^{\alpha}_{A}\dot{u}_\alpha - u_\alpha T^{\alpha\lambda}_{A}{}_{|\lambda} \ . \tag{5.59}$$

Now, let us note three thermodynamical relationships which are special cases of equation (A19):

$$\dot{I}_{A10} = J_{A10}\dot{\alpha}_A - J_{A20}\dot{\beta}_A \ ; \tag{5.60}$$

$$\dot{I}_{A20} = J_{A20}\dot{\alpha} - J_{A30}\dot{\beta}_A \quad ; \tag{5.61}$$

$$\dot{I}_{A30} = J_{A30}\dot{\alpha} - J_{A40}\dot{\beta}_A \quad . \tag{5.62}$$

Equations (5.60) and (5.61) may be used to solve for $\dot{\alpha}_A$ and $\dot{\beta}_A$:

$$\begin{bmatrix} \dot{\alpha}_A \\ \dot{\beta}_A \end{bmatrix} = \frac{1}{D_{A20}} \begin{bmatrix} J_{A30} & -J_{A20} \\ J_{A20} & -J_{A10} \end{bmatrix} \begin{bmatrix} \dot{I}_{A10} \\ \dot{I}_{A20} \end{bmatrix} \quad . \tag{5.63}$$

Substituting this solution into equation (5.62) tells us that

$$3\dot{I}_{A31} = U_{7A}\dot{I}_{A10} + U_{8A}\dot{I}_{A20} \quad ; \tag{5.64}$$

where we have defined

$$U_{7A} \equiv 3(J_{A30}J_{A31} - J_{A20}J_{A41})/D_{A20} \quad ; \tag{5.65}$$

$$U_{8A} \equiv 3(J_{A10}J_{A41} - J_{A20}J_{A31})/D_{A20} \quad . \tag{5.66}$$

When equations (5.58) and (5.59) are inserted into equation (5.64), we find that

$$\begin{aligned} 3\dot{I}_{A31} + 5I_{A31}\theta = & -I_{A31}\Omega_A\theta - U_{7A}\delta\dot{n}_A \\ & - U_{8A}\delta\dot{\rho}_A - (\, U_{8A} + U_{7A}/\eta_A \,)h^{\alpha}_{A|\alpha} \\ & + U_{7A}/\eta_A \; q^{\alpha}_{A|\alpha} - (\, U_{8A} + \tilde{\eta}_A U_{7A} \,)h^{\alpha}_A\dot{u}_\alpha \\ & + \tilde{\eta}_A U_{7A} \; q^{\alpha}_A\dot{u}_\alpha - U_{8A}u_\alpha T^{\alpha\lambda}_{A}{}_{|\lambda} \;\; ; \end{aligned} \tag{5.67}$$

where Ω_A is given by

$$\Omega_A = (I_{A10}U_{7A} + \eta_A I_{A10}U_{8A})/I_{A31} - 5 \;\; ; \tag{5.68}$$

and is identical to expression (4.35).

In equation (5.21) we may now substitute for the second term in parentheses via equation (5.67). Also, we may substitute in for the first term in parentheses via equation (5.58). Therefore equation (5.21) becomes the following expression:

$$\begin{aligned} u_\alpha u_\beta \overset{\circ}{U}{}^{\alpha\beta\gamma}_{A}{}_{|\alpha} = & - (1 + U_{7A})\delta\dot{n}_A - U_{8A}\delta\dot{\rho}_A - I_{A31}\Omega_A\theta \\ & - [U_{8A} + (1+U_{7A})/\eta_A]h^{\alpha}_{A|\alpha} \\ & + [(1+U_{7A})/\eta_A]q^{\alpha}_{A|\alpha} - [U_{8A}+\tilde{\eta}_A(1+U_{7A})]h^{\alpha}_A\dot{u}_\alpha \\ & + \tilde{\eta}_A(1 + U_{7A})q^{\alpha}_A\dot{u}_\alpha - U_{8A}u_\alpha T^{\alpha\lambda}_{A}{}_{|\lambda} \;\; . \end{aligned} \tag{5.69}$$

Let us now re-express equation (5.22). In equation (5.56) we solve for the $\dot{u}_\mu$ on the left-hand side:

$$\dot{u}_\mu = \frac{U_{4A}}{I_{A10}+5I_{A31}} \Big\{ \nabla_\mu I_{A21} + \frac{1}{3} \nabla_\mu \pi_A + (\delta p_A + \pi_A/3)\dot{u}_\mu + \dot{h}_A^{\lambda}\Delta_{\lambda\mu} + \pi_{A\mu}{}^{\lambda}{}_{|\lambda} - h_A{}^{\lambda}\omega_{\lambda\mu} - \Delta_{\mu\alpha} T_A^{\alpha\lambda}{}_{|\lambda} \Big\} . \tag{5.70}$$

This is one relation that we shall require. The other relation that we require is obtained by noting that the thermodynamical relationship (Israel [12])

$$(\Lambda_A + \eta_A^2) \, dI_{A21} - \eta_A \, dI_{A31} = J_{A21}\Lambda_A \, d\alpha_A , \tag{5.71}$$

may be rewritten by expressing the differentials as spatial derivatives:

$$\nabla_\mu I_{A31} = U_{4A}\nabla_\mu I_{A21} - (J_{A21}\Lambda_A/\eta_A)\nabla_\mu d\alpha_A \ . \tag{5.72}$$

We now substitute equations (5.70) and (5.72) directly into the right-hand side of equation (5.22). We eventually obtain the following expression:

$$- \Delta_{\mu\alpha} u_\beta \mathring{U}_A^{\alpha\beta\gamma}{}_{|\gamma} = - (J_{A21}\Lambda_A/\eta_A)\nabla_\mu \alpha_A - U_{4A}\Big\{ \frac{1}{3} \nabla_\mu \pi_A + (\delta \rho_A + \pi_A/3)\dot{u}_\mu + \Delta_{\mu\alpha}\dot{h}_A^{\alpha} - h_A^{\lambda}\omega_{\lambda\mu} + \pi_{A\mu|\lambda}^{\lambda} - \Delta_{\mu\alpha} T_A^{\alpha\lambda}{}_{|\lambda} \Big\}. \tag{5.73}$$

We now have all the information we need, in the appropriate form, to compute the transport equations.

E. The First Transport Equation

To obtain the first transport equation, we add equations (5.69) and (5.24). We then have that

$$u_\alpha u_\beta U_A^{\alpha\beta\gamma}|_\gamma + U_{8A} u_\alpha T_A^{\alpha\lambda}|_\lambda = - I_{A31}\Omega_A \Big\{ \theta + t_{A10}\delta\dot{n}_A + t_{A11}\delta\dot{\rho}_A + t_{A12}\dot{\pi}_A + t_{A13}h_A^\lambda|_\lambda + t_{A14}q_A^\lambda|_\lambda + t_{A15}h_A^\lambda\dot{u}_\lambda + t_{A16}q_A^\lambda\dot{u}_\lambda \Big\} \quad ; \tag{5.74}$$

where we have defined

$$t_{A10} \equiv (U_{1A} - U_{7A})/(I_{A31}\Omega_A) \quad ; \tag{5.75}$$

$$t_{A11} \equiv (U_{2A} - U_{8A})/(I_{A31}\Omega_A) \quad ; \tag{5.76}$$

$$t_{A12} \equiv U_{3A}/(I_{A31}\Omega_A) \quad ; \tag{5.77}$$

$$t_{A13} \equiv \Big\{ U_{4A} - U_{8A} - [(1+U_{7A})/\eta_A] \Big\} \div (I_{A31}\Omega_A) \quad ; \tag{5.78}$$

$$t_{A14} \equiv \Big\{ U_{5A} + [(1+U_{7A})/\eta_A] \Big\} \div (I_{A31}\Omega_A) \quad ; \tag{5.79}$$

$$t_{A15} \equiv \Big\{ 2U_{4A} + \bar{U}_{4A} - U_{8A} - \tilde{\eta}_A(1+U_{7A}) \Big\} \div (I_{A31}\Omega_A) \quad ; \tag{5.80}$$

$$t_{A16} \equiv \left\{ 2U_{5A} + \bar{U}_{5A} + \tilde{\eta}_A(1+U_{7A}) \right\} \div (I_{A31}\Omega_A) \quad . \tag{5.81}$$

The coefficients introduced here (and later on in our discussion) are figuratively of the form t_{Aij} . The indices i and j mean that t_{Aij} is the jth coefficient in the ith transport equation.

The left hand side of (5.74) must be evaluated via equation (2.21). Since this procedure is rather complex, the evaluation of the right hand side of equation (2.21) is deferred to chapter VII. From chapter VII, therefore, the left-hand side of equation (5.74) may be rewritten via equations (7.58) and (7.56). We therefore have that

$$\begin{aligned} u_\alpha u_\beta U_A^{\alpha\beta\lambda}|_\lambda + U_{8A} u_\alpha T_A^{\alpha\lambda}|_\lambda = {} & \chi_{A0} \\ & + \sum_B \left\{ \chi_{AB1}\delta n_B + \chi_{AB2}\delta\rho_B + \chi_{AB3}\pi_B \right\} \; ; \end{aligned} \tag{5.82}$$

where we have defined

$$\chi_{A0} \equiv \tilde{\chi}_{A0} - U_{8A}L_{A0} \quad ; \tag{5.83}$$

$$\chi_{ABi} \equiv \tilde{\chi}_{ABi} - U_{8A}L_{ABi} \quad , \quad i = 1, 2, 3. \tag{5.84}$$

Let us now replace the left-hand side of equation (5.74) by the right-hand side of equation (5.82). This gives us our first transport equation:

$$\chi_{A0} + \sum_B (\chi_{AB1}\delta n_B + \chi_{AB2}\delta\rho_B + \chi_{AB3}\pi_B)$$

$$= - I_{A31}\Omega_A \Big\{ \theta + t_{A10}\delta\dot{n}_A + t_{A11}\delta\dot{\rho}_A + t_{A12}\dot{\pi}_A \tag{5.85}$$

$$+ t_{A13}h^{\lambda}_{A|\lambda} + t_{A14}q^{\lambda}_{A|\lambda} + t_{A15}h^{\lambda}_{A}\dot{u}_{\lambda} + t_{A16}q^{\lambda}_{A}\dot{u}_{\lambda} \Big\} \;\; .$$

F. The Second Transport Equation

To obtain the second transport equation we add equations (5.73) and (5.25); we therefore have that

$$\Delta_{\mu\alpha}u_{\beta}U_A^{\alpha\beta\lambda}|_{\lambda} + U_{4A}\Delta_{\mu\alpha}T_A^{\alpha\lambda}|_{\lambda} = (J_{A21}\Lambda_A/\eta_A)\Big\{ \nabla_{\mu}\alpha_A$$

$$+ t_{A20}\nabla_{\mu}\delta n_A + t_{A21}\nabla_{\mu}\delta\rho_A + t_{A22}\nabla_{\mu}\pi_A \tag{5.86}$$

$$+ (t_{A23}\delta n_A + t_{A24}\delta\rho_A + t_{A25}\pi_A)\dot{u}_{\mu} + t_{A26}\Delta_{\mu}{}^{\lambda}\dot{q}_{A\lambda}$$

$$- t_{A26}q_A^{\lambda}\omega_{\lambda\mu} + t_{A27}\pi_{A\mu}{}^{\lambda}|_{\lambda} + t_{A28}\pi_{A\mu}{}^{\lambda}\dot{u}_{\lambda} \Big\} \;\; ;$$

where we have defined

$$t_{A20} \equiv - \frac{1}{3}\, U_{1A}/(J_{A21}\Lambda_A/\eta_A) \;\; ; \tag{5.87}$$

$$t_{A21} \equiv - \frac{1}{3}\, U_{2A}/(J_{A21}\Lambda_A/\eta_A) \;\; ; \tag{5.88}$$

$$t_{A22} \equiv - \frac{1}{3}\, (U_{4A} - U_{3A})/(J_{A21}\Lambda_A/\eta_A) \;\; ; \tag{5.89}$$

$$t_{A23} \equiv [1 - \frac{1}{3}(5U_{1A} + \bar{U}_{1A})]/(J_{A21}\Lambda_A/\eta_A) \quad ; \qquad (5.90)$$

$$t_{A24} \equiv [U_{4A} - \frac{1}{3}(5U_{2A} + \bar{U}_{2A})]/(J_{A21}\Lambda_A/\eta_A) \quad ; \qquad (5.91)$$

$$t_{A25} \equiv [\frac{1}{3}U_{4A} - \frac{1}{3}(5U_{3A} + \bar{U}_{3A})]/(J_{A21}\Lambda_A/\eta_A) \quad ; \qquad (5.92)$$

$$t_{A26} \equiv U_{5A}/(J_{A21}\Lambda_A/\eta_A) \quad ; \qquad (5.93)$$

$$t_{A27} \equiv (U_{4A} - U_{6A})/(J_{A21}\Lambda_A/\eta_A) \quad ; \qquad (5.94)$$

$$t_{A28} \equiv (U_{6A} + \bar{U}_{6A})/(J_{A21}\Lambda_A/\eta_A) \quad . \qquad (5.95)$$

The left-hand side of equation (5.86) may be rewritten via equations (7.57) and (7.59). Thus we have

$$\Delta_{\mu\alpha}u_{\beta}U_A^{\alpha\beta\gamma}|_{\gamma} + U_{4A}u_{\alpha}T_A^{\alpha\lambda}|_{\lambda} = \sum_B(\chi_{AB4}h_{B\mu} + \chi_{AB5}q_{B\mu}) \quad ; \qquad (5.96)$$

where we have defined

$$\chi_{ABi} \equiv \tilde{\chi}_{ABi} - U_{4A}L_{ABi} \, , \; i = 4, 5. \qquad (5.97)$$

Then, replacing the left-hand side of equation (5.86) by the right-hand side of equation (5.96), we deduce the second transport equation:

$$\sum_B (\chi_{AB4} h_{B\mu} + \chi_{AB5} q_{B\mu}) = \frac{J_{A21}\Lambda_A}{\eta_A}\Big\{\nabla_\mu \alpha_A + t_{A20}\nabla_\mu \delta n_A + t_{A21}\nabla_\mu \delta\rho_A + t_{A22}\nabla_\mu \pi_A + (t_{A23}\delta n_A + t_{A24}\delta\rho_A + t_{A25}\pi_A)\dot{u}_\mu + t_{A26}(\Delta_\mu^{\ \lambda}\dot{q}_{A\lambda} - q_A^{\ \lambda}\omega_{\lambda\mu}) + t_{A27}\pi_{A\mu}^{\ \ \ \lambda}{}_{|\lambda} + t_{A28}\pi_{A\mu}^{\ \ \ \lambda}\dot{u}_\lambda\Big\} . \tag{5.98}$$

G. The Third Transport Equation

The third transport equation is obtained by adding equations (5.23) and (5.26) which gives us an expression for $(\Delta_{\mu\alpha}\Delta_{\nu\beta} - \frac{1}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta})U_A^{\alpha\beta\gamma}{}_{|\gamma}$; however, this quantity is also given by equation (7.60). Equating these two expressions presents us with the third transport equation:

$$\sum_B \chi_{AB6}\pi_{B\mu\nu} = -I_{A31}\Big\{u_{<\mu|\nu>} + t_{A30}h_{A<\mu|\nu>} + t_{A31}q_{A<\mu|\nu>} + t_{A32}h_{A<\mu}\dot{u}_{\nu>} + t_{A33}q_{A<\mu}\dot{u}_{\nu>} + t_{A34}(\dot{\pi}_{A<\mu|\nu>} - 2\pi_{A<\mu}^{\ \ \ \ \lambda}\omega_{\lambda\nu>})\Big\} . \tag{5.99}$$

We may formally solve for the bulk stress, the heat flux, and the viscous stresses by noting that the transport equations are matrix equations in the species indices A and B. Then, if we assume that χ_{ABi} $(i=3,5,6)$ has a matrix inverse χ^{-1}_{ABi} $(i=3,5,6)$, we multiply the transport equations by these inverse matrices to obtain the following three versions of the transport equations:

$$\begin{aligned}\pi_A = -\sum_B \tilde{\nu}_{AB}\Big\{&\theta + t_{B10}\delta\dot{n}_B + t_{B11}\delta\dot{\rho}_B + t_{B12}\dot{\pi}_B \\ &+ t_{B13}h_B^{\lambda}{}_{|\lambda} + t_{B14}q_B^{\lambda}{}_{|\lambda} + t_{B15}h_B^{\lambda}\dot{u}_\lambda + t_{B16}q_B^{\lambda}\dot{u}_\lambda\Big\} \\ &- \sum_B \chi_{AB3}^{-1}\chi_{B0} - \sum_{BC}\sum(\chi_{AB3}^{-1}\chi_{BC1}\delta n_C + \chi_{AB3}^{-1}\chi_{BC2}\delta\rho_C) \quad ;\end{aligned} \tag{5.100}$$

$$\begin{aligned}q_{A\mu} = \sum_B \lambda_{AB}\Big\{&\nabla_\mu\alpha_B + t_{B20}\nabla_\mu\delta n_B + t_{B21}\nabla_\mu\delta\rho_B \\ &+ t_{B22}\nabla_\mu\pi_B + (t_{B23}\delta n_B + t_{B24}\delta\rho_B + t_{B25}\pi_B)\dot{u}_\mu \\ &+ t_{B26}(\Delta_\mu{}^{\lambda}\dot{q}_{B\lambda} - q_B^{\lambda}\omega_{\lambda\mu}) + t_{B27}\pi_{B\mu}{}^{\lambda}{}_{|\lambda} \\ &+ t_{B28}\pi_{B\mu}{}^{\lambda}\dot{u}_\lambda\Big\} - \sum_{BC}\sum \chi_{AB5}^{-1}\chi_{BC4}h_{C\mu} \quad ;\end{aligned} \tag{5.101}$$

$$\begin{aligned}\pi_{A\mu\nu} = -\sum_B \nu_{AB}\Big\{&u_{<\mu|\nu>} + t_{B30}h_{B<\mu|\nu>} \\ &+ t_{B31}q_{B<\mu|\nu>} + t_{B32}h_{B<\mu}\dot{u}_{\nu>} + t_{B33}q_{B<\mu}\dot{u}_{\nu>} \\ &+ t_{B34}\dot{\pi}_{B<\mu\nu>} - 2t_{B34}\pi_{B<\mu}{}^{\lambda}\omega_{\lambda\nu>}\Big\} \quad ;\end{aligned} \tag{5.102}$$

where we have defined the bulk viscosity matrix, the heat conductivity matrix, and the viscosity matrix respectively by

$$\tilde{\nu}_{AB} \equiv \chi_{AB3}^{-1}I_{B31}\Omega_B \quad ; \tag{5.103}$$

$$\lambda_{AB} \equiv \chi^{-1}_{AB5} J_{B21} \Lambda_B / n_B \quad ; \qquad (5.104)$$

$$\nu_{AB} \equiv \chi^{-1}_{AB6} I_{B31} \quad . \qquad (5.105)$$

The three transport equations above reduce, for a single species gas, to the forms reported by Israel and Stewart [16] provided we apply the fitting conditions $\delta n_A = \delta \rho_A = 0$ and choose u^α by setting $h^\alpha_A = 0$. We also note that the transport equations are linear, first order, coupled differential equations which in the single species case are hyperbolic and lead to propagation speeds of viscous and thermal effects less than that of light (Stewart[34]). This suggests that for the multi-component gas the transport equations are hyperbolic and describe causal propagations of the heat fluxes and viscous stresses; however, a proof of this is beyond the scope of this thesis. Our discussion of the massive components of the gas is complete. We now turn our attention to the light-like components which we will discuss in chapter VI.

VI. Massless Particles

Massless particles, such as photons or neutrinos, possess zero length four velocities, that is $w_{L\alpha}w_L^\alpha=0$ (we reserve the symbol L for massless particles). This condition immediately implies that the energy-momentum tensor for these particles, as defined by equation (2.16), is trace free. We have already noted in chapter IV that this result deprives us of one of the conditions necessary to obtain a unique solution of the non-equilibrium situation, for massless particles, in the Grad fourteen moment approximation; this is because the number of linear equations relating fourteen variables is reduced from fourteen to thirteen in the case of massless particles. Hence, we must modify the fourteen moment relativistic Grad method in an appropriate manner to obtain such a solution.

In this chapter we shall discuss precisely how to modify the fourteen moment method to obtain the non-equilibrium solution for massless particles. To achieve this we shall present our discussion as a miniature version of the previous chapters on massive particles. We will start out by specifying how the Boltzmann equation must be modified to account for the interactions between massless and massive particles. We introduce a change of variables to facilitate subsequent calculations by expressing the massless particle's four-momentum in terms of a frequency and a unit spatial vector, both of which are defined with respect to the flow vector u^α. This change of variable allows us to describe the physics in a frequency dependent format. The equilibrium distribution function for massless particles allows us to define frequency dependent integrals and thermodynamic functions which will be employed to analyse the non-equilibrium form of the distribution function. This distribution function is calculated via a frequency dependent version of the relativistic Grad method; this is obtained by assuming that the coefficients in the power series expansion in the particle's four-momentum depends on frequency as well as on position. The non-equilibrium form of the distribution function, when inserted into the Boltzmann equation, leads directly to the transport equations. Finally, we shall derive the entropy production and define the coefficients of bulk viscosity, shear viscosity, and heat flux.

A. Creation and Annihilation Processes

Let us begin our analysis by considering the Boltzmann equation for massless particles. The collision term, $D_{coll}N_L$, as stated in chapter II, only takes into account binary collisions. Binary collisions, by themselves, do not adequately describe massless particles. For example, the number of photons in an equilibrium state is a function of the temperature, that is, the number of photons from state to state is not conserved. We therefore introduce into our description of the gas creation and annihilation processes in which the number of massless particles is not conserved

[1]. These processes are figuratively of the form $A+B+L \rightarrow A^*+B^*$ (annihilation or absorption) and $A+B \rightarrow A^*+B^*+L$ (creation).

We describe annihilation and creation processes for massless particles by the transition probabilities $W(p_A,p_B,p_L|\overset{*}{p}_A,\overset{*}{p}_B)$ and $W(p_A,p_B|\overset{*}{p}_A,\overset{*}{p}_B,p_L)$ respectively. Then, following our analysis of chapter II, the contribution to $D_{coll}N_L$ by these processes is given by

$$\begin{aligned} T(N_L) = &\sum_{AB}\sum \int N_A N_B \overset{*}{\Delta}_A \overset{*}{\Delta}_B \Delta_L W(p_A,p_B|\overset{*}{p}_A,\overset{*}{p}_B,p_L)\, d^4V \\ &-\sum_{AB}\sum \int N_A N_B N_L \overset{*}{\Delta}_A \overset{*}{\Delta}_B W(p_A,p_B,p_L|\overset{*}{p}_A,\overset{*}{p}_B)\, d^4V \quad ; \end{aligned} \tag{6.1}$$

where $d^4V \equiv dV_A dV_B d\overset{*}{V}_A d\overset{*}{V}_B$ Then the Boltzmann equation for massless particles is given by

$$N_L{}_{||\alpha} w^\alpha_L = D_{coll}N_L = B(N_L) + T(N_L) \quad ; \tag{6.2}$$

where $B(N_L)$, the contribution due to binary collisions, is given by the right hand side of equation (2.5) for A=L and the sum is taken over all massive particles. We are assuming here that interactions between massless particles are negligible (e.g. photon-photon and photon-neutrino processes).

For the sake of consistency, we should also add the contribution of creation and absorption processes to $D_{coll}N_A$ for massive particles. This contribution is given by [1,12]

$$\begin{aligned}\overset{*}{T}(N_A) = &\sum_{BL}\sum \int \overset{*}{N}_A \overset{*}{N}_B N_L \Delta_A \Delta_B W(\overset{*}{p}_A,\overset{*}{p}_B,p_L|p_A,p_B)dV_B d\overset{*}{V}_A d\overset{*}{V}_B dV_L \\ &-\sum_{BL}\sum \int N_A N_B N_L \overset{*}{\Delta}_A \overset{*}{\Delta}_B W(p_A,p_B,p_L|\overset{*}{p}_A,\overset{*}{p}_B)dV_B d\overset{*}{V}_A d\overset{*}{V}_B dV_L \\ &+\sum_{BL}\sum \int \overset{*}{N}_A \overset{*}{N}_B \Delta_A \Delta_B \Delta_L W(\overset{*}{p}_A,\overset{*}{p}_B|p_A,p_B,p_L)dV_B d\overset{*}{V}_A d\overset{*}{V}_B dV_L \\ &-\sum_{BL}\sum \int N_A N_B \overset{*}{\Delta}_A \overset{*}{\Delta}_B \Delta_L W(p_A,p_B|\overset{*}{p}_A,\overset{*}{p}_B,p_L)dV_B d\overset{*}{V}_A d\overset{*}{V}_B dV_L \ .\end{aligned} \tag{6.3}$$

Then the Boltzmann equation for massive particles is

$$N_A||_\alpha w_A^\alpha = D_{coll} N_A = B(N_A) + \overset{*}{T}(N_A) \quad ; \tag{6.4}$$

where $B(N_A)$ is given by equation (2.5).

The master balance equation (2.7) retains its validity here. We just have to ensure that $D_{coll}N$ includes $T(N_L)$ or $\overset{*}{T}(N_A)$. As a consequence of these extra terms, the master balance equation implies that the massless particle number (mass) flux is not conserved but the number and mass fluxes for massive particles are conserved. Furthermore, the energy-momentum tensor of the whole gas is still conserved and the Boltzmann H theorem is still valid [12]. Then the equilibrium situation as described in chapter II remains valid. The equilibrium restrictions on the thermal potentials α_A require $\alpha_L=0$ if we include creation and annhilation processes. Hence, for massless particles we shall take $\alpha_L = 0$.

B. Spectral Analysis

Let us begin the technical discussion of massless particles by defining the frequency ν_L (or energy in units such that h=1) and the spatial vector k^α_L by

$$\nu_L \equiv - u_\alpha w^\alpha_L \ ; \ k^\alpha_L \equiv \Delta^\alpha{}_\lambda w^\lambda_L / \nu_L \ ; \tag{6.5}$$

so that $k^\alpha_L k_{L\alpha} = 1$. In terms of ν_L , k^α_L , and u^α we decompose the null-like four-momentum by

$$w^\alpha_L = \nu_L (u^\alpha + k^\alpha_L) \quad . \tag{6.6}$$

In a local Lorentz frame

$$g_{\alpha\beta} \simeq \eta_{\alpha\beta} \ ; \ u^\alpha = \delta^\alpha_4 \ ; \ k^\alpha_L = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta, 0), \tag{6.7}$$

the four dimensional volume dV_L may be written as

$$dV_L = \nu_L d\nu_L d\Omega_L \ ; \ d\Omega_L \equiv \sin\theta d\theta d\phi \ ; \tag{6.8}$$

where $d\Omega_L$ is the differential of solid angle. Integration of an odd number of k^α_L's yields zero whereas integration over an even number of k^α_L's does not [1,2]. In particular, we have that

$$\int k^\alpha_L d\Omega_L = \int k^\alpha_L k^\beta_L k^\gamma_L d\Omega_L = 0 \ ; \ \int k^\alpha_L k^\beta_L d\Omega_L = \frac{4\pi}{3} \Delta^{\alpha\beta} \ ;$$
$$\int k^\alpha_L k^\beta_L k^\mu_L k^\nu_L d \ {}_L = \frac{4\pi}{15} (\Delta^{\alpha\beta}\Delta^{\mu\nu} + \Delta^{\alpha\mu}\Delta^{\beta\nu} + \Delta^{\alpha\nu}\Delta^{\beta\mu}) \quad . \tag{6.9}$$

Spectral (frequency dependent) forms of the number flux and energy-momentum tensor may be defined as

$$N^{\alpha}_{L}(\nu_L) \equiv \int N_L w^{\alpha}_{L} \nu_L d\Omega_L \quad ; \tag{6.10}$$

$$T^{\alpha\beta}_{L}(\nu_L) \equiv \int N_L w^{\alpha}_{L} w^{\beta}_{L} \nu_L d\Omega_L \quad . \tag{6.11}$$

When we integrate over frequency ν_L we just obtain the usual number flux and energy-momentum tensor:

$$N^{\alpha}_{L} = \int N^{\alpha}_{L}(\nu_L) d\nu_L \ ; \ T^{\alpha\beta}_{L} = \int T^{\alpha\beta}_{L}(\nu_L) d\nu_L \quad . \tag{6.12}$$

We may extend the decompositions (2.29) and (2.31) to the spectral case:

$$\begin{aligned} N^{\alpha}_{L}(\nu_L) &= n_L(\nu_L) u^{\alpha} + j^{\alpha}_{L}(\nu_L) \quad ; \\ T^{\alpha\beta}_{L}(\nu_L) &= \rho_L(\nu_L) u^{\alpha} u^{\beta} + \frac{1}{3} \rho_L(\nu_L) \Delta^{\alpha\beta} \\ &\quad + h^{\alpha}_{L}(\nu_L) u^{\beta} + h^{\beta}_{L}(\nu_L) u^{\alpha} + \pi^{\alpha\beta}_{L}(\nu_L) \quad ; \end{aligned} \tag{6.13}$$

where we have incorporated the requirement that $T^{\lambda}_{L\lambda}(\nu_L) = 0$. Here the spectral number density, particle drift, energy density, momentum flux, and viscous stresses are given by $n_L(\nu_L)$, $j^{\alpha}_{L}(\nu_L)$, $\rho_L(\nu_L)$, $h^{\alpha}_{L}(\nu_L)$, and $\pi^{\alpha\beta}_{L}(\nu_L)$ respectively.

In equilibrium, $N_L = \overset{\circ}{N}_L$, so that we extend the definitions (3.5) to the frequency dependent case:

$$I_L(\nu_L)^{\alpha_1\cdots\alpha_n} \equiv \int \mathring{N}_L w_L^{\alpha_1}\cdots w_L^{\alpha_n}\,\nu_L d\Omega_L \quad ;$$
$$J_L(\nu_L)^{\alpha_1\cdots\alpha_n} \equiv \int (\mathring{N}_L\mathring{A}_L/g_L) w_L^{\alpha_1}\cdots w_L^{\alpha_n}\,\nu_L d\Omega_L \quad . \tag{6.14}$$

Then, integration over the frequency just produces equation (3.5). Applying the analysis of chapter III to this case tells us that the integrals (6.14) have the structures (3.9) and (3.10) provided we replace I_{Lnq} and J_{Lnq} by their spectral forms:

$$I_{Lnq}(\nu_L) = \frac{(-1)^n}{(2q+1)!!} I_L(\nu_L)^{\alpha(n)}\Pi_{(q)\alpha(n)} \quad ;$$
$$J_{Lnq}(\nu_L) = \frac{(-1)^n}{(2q+1)!!} J_L(\nu_L)^{\alpha(n)}\Pi_{(q)\alpha(n)} \quad ; \tag{6.15}$$

Expressions for these coefficients are produced in Appendix A.

In equilibrium we therefore have

$$N_L^\alpha(\nu_L) = I_L(\nu_L)^\alpha \;;\; T_L^{\alpha\beta}(\nu_L) = I_L(\nu_L)^{\alpha\beta} \;. \tag{6.16}$$

Comparing these results with (6.13) now requires that, for equilibrium,

$$n_L(\nu_L) = I_{L10}(\nu_L) \;,\; j_L^\alpha(\nu_L) = 0 \quad ;$$
$$\rho_L(\nu_L) = I_{L20}(\nu_L) \;,\; h_L^\alpha(\nu_L) = 0 \;,\; \pi_L^{\alpha\beta}(\nu_L) = 0 \;. \tag{6.17}$$

C. The Modified Grad Method and Non-equilibrium

We may now consider the description of the deviations from equilibrium. We shall assume that the flow vector u^α has been chosen via frame changes to be common to all species. We expand $\ln(N_L/\Delta_L)$, as in the massive particle case, by

$$\ln(N_L/\Delta_L) = \ln(\overset{\circ}{N}_L/\overset{\circ}{\Delta}_L) + f_L \quad ; \qquad (6.18)$$

where we assume that f_L is given by

$$f_L \equiv \tilde{a}_L(x,\nu_L) + \tilde{b}^{\lambda}_L(x,\nu_L)w_{L\lambda} + \tilde{c}^{\lambda\tau}_L(x,\nu_L)w_{L\lambda}w_{L\tau} \quad ; \qquad (6.19)$$

where $\tilde{c}^{\lambda\tau}_L$ may be assumed to be trace free. The crucial assumption here is that the coefficients $\tilde{a}_L$, $\tilde{b}^{\lambda}_L$, and $\tilde{c}^{\lambda\tau}_L$ which appear here are also functions of frequency as well as of position, whereas in the massive case (Grad fourteen moment approximation) they were assumed to be functions of position alone.

If we apply the decomposition (6.6) for w^α_L in equation (6.19) we obtain

$$f_L = a_L(x,\nu_L) + b^{\lambda}_L(x,\nu_L)k_{L\lambda} + c^{\lambda\tau}_L(x,\nu_L)k_{L\lambda}k_{L\tau} \quad ; \qquad (6.20)$$

where we have defined

$$a_L(x,\nu_L) \equiv \tilde{a}_L + \nu_L \tilde{b}_L^{\lambda} u_\lambda + \frac{4}{3}\nu_L^2\, \tilde{c}_L^{\lambda\tau} u_\lambda u_\tau \quad ;$$

$$b_L^{\alpha}(x,\nu_L) \equiv \nu_L \tilde{b}_L^{\lambda} \Delta_\lambda{}^{\alpha} + 2\nu_L^2 \tilde{c}_L^{\lambda\tau} u_\lambda \Delta_\tau{}^{\alpha} \quad ; \tag{6.21}$$

$$c_L^{\alpha\beta}(x,\nu_L) \equiv \nu_L^2\, (\Delta^{\alpha}{}_{\lambda}\Delta^{\beta}{}_{\tau} - \frac{1}{3}\Delta^{\alpha\beta}\Delta_{\lambda\tau})\tilde{c}_L^{\lambda\tau} \quad ;$$

which now become our primary variables appearing in f_L . The decomposition (6.20) is essentially a spherical harmonic expression of f_L [1,2]. The number of unknown variables herein is nine as compared to fourteen in the massive case.

In terms of f_L we may write N_L as

$$N_L = \mathring{N}_L + \delta N_L \;;\; \delta N_L = (\mathring{N}_L \mathring{\Delta}_L / g_L) f_L = \frac{J_{L20}(\nu_L)}{4\pi\nu_L^2}\, f_L \quad . \tag{6.22}$$

When we insert this expression along with (6.20) into (6.11) and employ definitions (6.14) and (6.15) we obtain an expression for the non-equilibrium spectral energy-momentum tensor. Comparison of this result with the decomposition (6.13) tells us that

$$\rho_L(\nu_L) = I_{L20}(\nu_L) + \delta\rho_L(\nu_L) \;;\; \delta\rho_L(\nu_L) = J_{L20}(\nu_L) a_L \;;$$

$$h_L^{\alpha}(\nu_L) = \frac{1}{3} J_{L20}(\nu_L) b_L^{\alpha} \;;\; \pi_L^{\alpha\beta}(\nu_L) = \frac{2}{15} J_{L20}(\nu_L) c_L^{\alpha\beta} \quad . \tag{6.23}$$

Solving equation (6.23) for a_L , b_L^{α}, and $c_L^{\alpha\beta}$ we find for f_L the following expression:

$$f_L = \frac{1}{J_{L20}(\nu_L)} \left\{ \delta\rho_L(\nu_L) + 3h_L^{\alpha}(\nu_L) k_{L\alpha} + \frac{15}{2}\, \pi_L^{\alpha\beta}(\nu_L) k_{L\alpha} k_{L\beta} \right\} . \tag{6.24}$$

Therefore, the deviation from equilibrium is completely specified by the spectral energy-momentum tensor.

The spectral number flux may be evaluated in terms of equations (6.22) and (6.24); comparing the result with expression (6.13) gives us

$$\begin{aligned} &n_L(\nu_L) = I_{L10}(\nu_L) + \delta n_L(\nu_L) \quad ; \\ &\delta n_L(\nu_L) = [J_{L10}(\nu_L)/J_{L20}(\nu_L)]\delta\rho_L(\nu_L) = \delta\rho_L(\nu_L)/\nu_L \quad ; \\ &j^{\alpha}_L(\nu_L) = [J_{L10}(\nu_L)/J_{L20}(\nu_L)]h^{\alpha}_L(\nu_L) = h^{\alpha}_L(\nu_L)/\nu_L \quad . \end{aligned} \tag{6.25}$$

We shall assume herein that $\delta n_L(\nu_L)$, $\delta\rho_L(\nu_L)$, $j^{\alpha}_L(\nu_L)$, $h^{\alpha}_L(\nu_L)$, and $\pi^{\alpha\beta}_L(\nu_L)$ become zero for $\nu_L=0$ or $\nu_L=\infty$.

Integration of the structures (6.23) and (6.25) over frequency produces the structures (2.29) and (2.31) for the number flux and energy-momentum tensor where we define

$$\begin{aligned} &\rho_L \equiv \int \rho_L(\nu_L)d\nu_L \;;\; \delta\rho_L \equiv \int \delta\rho_L(\nu_L)d\nu_L \quad ; \\ &n_L \equiv \int n_L(\nu_L)d\nu_L \;;\; \delta n_L \equiv \int \delta n_L(\nu_L)d\nu_L \quad ; \\ &j^{\alpha}_L \equiv \int j^{\alpha}_L(\nu_L)d\nu_L \;;\; h^{\alpha}_L \equiv \int h^{\alpha}_L(\nu_L)d\nu_L \;;\; \pi^{\alpha\beta}_L \equiv \int \pi^{\alpha\beta}_L(\nu_L)d\nu_L \quad . \end{aligned} \tag{6.26}$$

The invariance of (6.18) and (6.22) under the fitting changes

$$\delta\alpha_L = 0 \; (\alpha_L = 0) \; , \; \beta_L = \beta'_L + \delta\beta_L \quad , \tag{6.27}$$

requires that

$$\delta\rho'_L(\nu_L) = \delta\rho_L(\nu_L) - J_{L30}(\nu_L)\delta\beta_L \quad ; \tag{6.28}$$

which in turn requires, from equation (6.25), that

$$\delta n_L'(\nu_L) = \delta n_L(\nu_L) - J_{L20}(\nu_L)\delta\beta_L \quad . \tag{6.29}$$

Integration of these two relationships over frequency and employing definitions (6.26) produces the usual fitting relationships (4.98). Although we may choose β_L such that $\delta n_L = \delta\rho_L = 0$, no choice of β_L can make $\delta n_L(\nu_L)$ or $\delta\rho_L(\nu_L)$ equal to zero.

For each massless component of the gas we define the radiation temperature T_{LR} , $\beta_{LR} = 1/kT_{LR}$ to be given by the fitting conditions $\delta n_L=\delta\rho_L=0$. We consider the matter part of the gas to consist only of massive particles. Therefore we define the matter temperature T_M , $\tilde{\beta}_M = 1/kT_M$ to be given by the fitting conditions $\delta n_A=0$, $\sum_A\delta\rho_A=0$, where the sum is taken over all massive particle species. This case was previously discussed in chapter IV. We also define the Eckart temperature T_E, $\tilde{\beta}_E = 1/kT_E$ by the fitting conditions

$$\sum_A \delta n_A + \sum_L \delta n_L = 0 \ ; \ \sum_A \delta\rho_A + \sum_L \delta\rho_L = 0 \quad . \tag{6.30}$$

These conditions are the generalization of the Eckart fitting conditions (Weinberg [41], Eckart [10]) $\delta n = \delta\rho = 0$ applied to the whole gas. The relationship between the Eckart temperature, the matter temperature, and the radiation temperatures may be found via fitting changes. To show that the conditions (6.30) are realizable we choose

$$\begin{aligned} \delta\alpha_1 &= -\sum_L [\delta n_L(\beta_L) - J_{L20}\delta\beta_L] \ , A=1 \ ; \\ \delta\alpha_A &= -(J_{A20}/J_{A10})\delta\beta_A \ , A \neq 1 \ ; \\ \delta\beta_L &= \beta_L - \beta_E \ ; \ \delta\beta_A = \beta_M - \beta_E \ . \end{aligned} \tag{6.31}$$

Then the first of the conditions in (6.30) is satisfied. We employ (6.31) to express $\delta\rho_A$, $\delta\rho_L$ in terms of $\delta\beta_A$ and $\delta\beta_L$:

$$\begin{aligned} \delta\rho_1(\beta_E) &= \delta\rho_1(\beta_M) - \frac{1}{J_{(1)20}} \sum_L [\delta n_L(\beta_L) - J_{L20}\delta\beta_L], \ A = 1 \ ; \\ \delta\rho_A(\beta_E) &= \delta\rho_A(\beta_M) - (D_{A20}/J_{A10})\delta\beta_A \ , A \neq 1 \ ; \\ \delta\rho_L(\beta_E) &= \delta\rho_L(\beta_L) - J_{L30}\delta\beta_L \ . \end{aligned} \tag{6.32}$$

The second restriction in (6.30) now implies that

$$\begin{aligned} \tilde{\beta}_E = \Big\{ & \beta_M \sum_{A\neq 1} D_{A20}/J_{A10} - \frac{1}{J_{(1)20}} \sum_L J_{L20}\beta_L + \frac{1}{J_{(1)20}} \sum_L \delta n_L \\ & - \sum_L (\delta\rho_L - J_{L30}\beta_L) \Big\} \div \Big\{ \sum_{A\neq 1} D_{A20}/J_{A10} + \sum_L (J_{L30} - \frac{1}{J_{(1)20}} J_{L20}) \Big\} . \end{aligned} \tag{6.33}$$

If there are no massless particles in the gas, then $T_E = T_M$.

D. The Spectral Transport Equations

Let us now derive the transport equations for massless particles. Our procedure shall consist of examining the conditions imposed by the Boltzmann equation on the deviations from equilibrium. These conditions will eventually produce the transport equations, in spectral form, and may be integrated to produce the actual transport equations. Before delving into this, let us discuss some preliminaries.

Let us differentiate ν_L and k_L^α as given by equation (6.5). This provides us with two very useful results

$$\nu_{L||\alpha} = -\nu_L k_L^\lambda u_{\lambda|\alpha} \quad ; \tag{6.34}$$

$$k_{L\alpha||\mu} = -u_{\alpha|\mu} + k_L^\lambda u_{\lambda|\mu}(u_\alpha + k_{L\alpha}) \quad . \tag{6.35}$$

Contraction with w_L^α produces

$$\nu_{L||\alpha} w_L^\alpha = -\nu_L^2 \left\{ \frac{\theta}{3} + k_L^\lambda \dot{u}_\lambda + \sigma_{\alpha\beta} k_L^\alpha k_L^\beta \right\} \quad ; \tag{6.36}$$

$$\begin{aligned} k_{L\alpha||\mu} w_L^\mu = \nu_L \Big\{ & -\dot{u}_\alpha + \frac{\theta}{3} u_\alpha - \omega_{\alpha\lambda} k_L^\lambda - \sigma_{\alpha\lambda} k_L^\lambda \\ & + (k_L^\lambda \dot{u}_\lambda + \sigma_{\mu\nu} k_L^\mu k_L^\nu)(u_\alpha + k_{L\alpha}) \Big\} \quad . \end{aligned} \tag{6.37}$$

Let χ_L be an arbitrary function of position and momentum, $\chi_L = \chi_L(x, w_L^\alpha)$. Equation (6.5) for ν_L and k_L^α constitute a change of variable, so that we may write $\chi_L = \chi_L(x, \nu_L, k_L^\alpha)$. Differention of χ_L holding momentum fixed by parallel propagation may be written in terms of these variables as [2]

$$\chi_{L||\mu} = \chi_{L;\mu} + \left(\frac{\partial \chi_L}{\partial \nu_L}\right) \nu_{L||\mu} + \left(\frac{\partial \chi_L}{\partial k_L^\alpha}\right) k^\alpha_{L||\mu} \quad ; \tag{6.38}$$

where the semi-colon means covariant differentiation holding

ν_L and k_L^α fixed. If χ_L is a function only of position and frequency, then the last term in equation (6.38) is zero. In this case, diffentiation holding frequency and k_L^α fixed has the useful property that

$$\int \chi_{L;\alpha} d\nu_L d\Omega_L = \int 4\pi\, \chi_{L;} d\nu_L = 4\pi \left\{ \int \chi_L d\nu_L \right\}_{|\alpha} \quad . \qquad (6.39)$$

In terms of the semi-colon operator, we define the spectral time and spatial derivatives by

$$\overset{\circ}{\chi}_L \equiv \chi_{L;\alpha} u^\alpha \quad ; \quad \overset{\circ}{\nabla}_\mu \chi_L \equiv \chi_{L;\lambda} \Delta^\lambda{}_\mu \quad ; \qquad (6.40)$$

which produce upon integration over frequency, via relationship (6.39), the usual time and spatial derivative as defined by equation (5.2).

To facilitate computation, we shall denote

$$\chi'_L \equiv (\partial \chi_L / \partial \nu_L)_{x,k_L^\alpha} \quad . \qquad (6.41)$$

Provided χ_L is zero for $\nu_L = \infty$ and $\nu_L = 0$ we have the following property:

$$- \int \chi'_L \nu_L d\nu_L = \int \chi_L d\nu_L \quad . \qquad (6.42)$$

Combining equations (6.22) and (6.25) gives us a convenient expression for the distribution function:

$$N_L = \frac{\rho_L(\nu_L)}{4\pi\nu_L^3} + \frac{1}{4\pi\nu_L^3}\left\{3h_L^{\lambda}(\nu_L)k_{L\lambda} + \frac{15}{2}\,\pi_L^{\lambda\tau}(\nu_L)k_{L\lambda}k_{L\tau}\right\} . \tag{6.43}$$

To obtain an expression for the left hand side of the Boltzmann equation (6.2) we differentiate (6.43) and contract it with w_L^{α} :

$$\begin{aligned} N_L||_{\alpha}w_L^{\alpha} &= [\rho_L(\nu_L)/(4\pi\nu_L^3]||_{\alpha}w_L^{\alpha} \\ &+ \frac{1}{4\pi}(\nu_L^{-3})||_{\alpha}w_L^{\alpha}\left\{3h_L^{\lambda}(\nu_L)k_{L\lambda} + \frac{15}{2}\pi_L^{\lambda\tau}(\nu_L)k_{L\lambda}k_{L\tau}\right\} \\ &+ \frac{3}{4\pi\nu_L^3}\left\{k_{L\lambda}h_L^{\lambda}(\nu_L)||_{\alpha}w_L^{\alpha} + h_L^{\lambda}(\nu_L)k_{L\lambda}||_{\alpha}w_L^{\alpha}\right\} \\ &+ \frac{15}{8\pi\nu_L^3}\left\{k_{L\mu}k_{L\lambda}\pi_L^{\mu\lambda}(\nu_L)||_{\alpha}w_L^{\alpha} + 2\pi_L^{\lambda\tau}(\nu_L)k_{L\lambda}k_{L\tau}||_{\alpha}w_L^{\alpha}\right\} . \end{aligned} \tag{6.44}$$

To proceed further, we shall re-express (6.44) term by term and then assemble the resulting expressions to find the desired expression for equation (6.44).

To compute the first two terms, it is helpful to know that

$$(\nu_L^{-3})||_{\alpha}w_L^{\alpha} = \frac{1}{\nu_L^3}\left\{\theta + k_L^{\lambda}(3\dot{u}_{\lambda}) + k_L^{\alpha}k_L^{\beta}(\tfrac{3}{2}\,u_{<\alpha|\beta>})\right\} . \tag{6.45}$$

With the help of this expression and equations (6.38), (6.40), and (6.41), the first term in equation (6.44) is given by

$$\begin{aligned} [\rho_L(\nu_L)/(4\pi\nu_L^3)]||_{\alpha}w_L^{\alpha} &= \frac{1}{4\pi\nu_L^3}\left\{\{\overset{\circ}{\rho}_L(\nu_L) + \tfrac{1}{3}[3\rho_L(\nu_L) - \nu_L\rho_L'(\nu_L)]\theta\}\right. \\ &+ k_L^{\alpha}\{\overset{\circ}{\nabla}_{\alpha}\rho_L(\nu_L) + [3\rho_L(\nu_L) - \nu_L\rho_L'(\nu_L)]\dot{u}_{\alpha}\} \\ &\quad k_L^{\alpha}k_L^{\beta}\{3\rho_L(\nu_L) - \nu_L\rho_L'(\nu_L)\}\,\tfrac{1}{2}\,u_{<\alpha|\beta>} \quad . \end{aligned} \tag{6.46}$$

The second term becomes, with the aid of equation (6.45),

$$\frac{1}{4\pi}(\nu_L^{-3})||_\alpha w_L^\alpha\left\{ 3h_L^\lambda(\nu_L)k_{L\lambda} + \frac{15}{2}\,\pi_L^{\lambda\tau}(\nu_L)k_{L\lambda}k_{L\tau}\right\}$$
$$= \frac{1}{4\pi\nu_L^2}\left\{ k_L^\alpha k_L^\beta[9h_{L\alpha}(\nu_L)\dot{u}_\beta] + k_L^\alpha k_L^\beta k_L^\gamma[\,\frac{45}{2}\,\pi_{L\alpha\beta}(\nu_L)\dot{u}_\gamma]\right\}. \tag{6.47}$$

For the remaining terms in equation (6.44) we obtain

$$k_L^\lambda h_{L\lambda}(\nu_L)||_\alpha w_L^\alpha = \nu_L\left\{ k_L^\lambda[\mathring{h}_{L\lambda}(\nu_L)]\right.$$
$$\left. + k_L^\alpha k_L^\beta[\,h_{L\alpha}(\nu_L)_{;\beta} - \nu_L h'_{L\alpha}(\nu_L)\dot{u}_\beta]\right\}\quad ; \tag{6.48}$$

$$h_L^\lambda(\nu_L)k_{L\lambda}||_\alpha w_L^\alpha = \nu_L\left\{[\,-\,h_L^\lambda(\nu_L)\dot{u}_\lambda]\right.$$
$$\left. + k_L^\lambda[\,-\,h_L^\alpha(\ddot{\nu}_L)\omega_{\alpha\lambda}] + k_L^\alpha k_L^\beta[\,h_{L\alpha}(\nu_L)\dot{u}_\beta]\right\}\quad ; \tag{6.49}$$

$$k_{L\alpha}k_{L\beta}\pi_L^{\alpha\beta}(\nu_L)||_\lambda w_L^\lambda = \nu_L\left\{ k_L^\alpha k_L^\beta[\,\mathring{\pi}_{L\alpha\beta}(\nu_L)\,]\right.$$
$$\left. + k_L^\alpha k_L^\beta k_L^\gamma[\,\pi_{L\alpha\beta}(\nu_L)_{;\gamma} - \nu_L\pi'_{L\alpha\beta}(\nu_L)\dot{u}_\gamma]\right\}\quad ; \tag{6.50}$$

$$k_{L\alpha}\pi_L^{\alpha\beta}(\nu_L)k_{L\beta}||_\lambda w_L^\lambda = \nu_L\left\{ k_L^\alpha[\,-\,\pi_{L\alpha}{}^\lambda(\nu_L)\dot{u}_\lambda]\right.$$
$$\left. + k_L^\alpha k_L^\beta[\,-\pi_{L\alpha}{}^\lambda(\nu_L)\omega_{\lambda\beta}] + k_L^\alpha k_L^\beta k_L^\gamma[\,\pi_{L\alpha\beta}(\nu_L)\dot{u}_\gamma]\right\}\quad . \tag{6.51}$$

We may now substitute equations (6.46) to (6.51) into (6.44). This procedure produces the required expression for

(6.44):

$$4\pi\nu_L^2 N_L||_\alpha w_L^\alpha = D_L + D_{L\alpha}k_L^\alpha + D_{L\alpha\beta}k_L^\alpha k_L^\beta + D_{L\alpha\beta\gamma}k_L^\alpha k_L^\beta k_L^\gamma \quad ; \qquad (6.52)$$

where we have set

$$D_L \equiv \overset{\circ}{\rho}_L(\nu_L) + \frac{1}{3}\{3\rho_L(\nu_L)-\nu_L\rho_L'(\nu_L)\}\theta - 3h_L^\lambda(\nu_L)\dot{u}_\lambda \quad ; \qquad (6.53)$$

$$\begin{aligned} D_{L\alpha} \equiv\ & \overset{\circ}{\nabla}_\alpha\rho_L(\nu_L) + \{3\rho_L(\nu_L)-\nu_L\rho_L'(\nu_L)\}\dot{u}_\alpha + 3\overset{\circ}{h}_{L\alpha}(\nu_L) \\ & - 3h_L^\lambda(\nu_L)\omega_{\lambda\alpha} - 15\,\pi_{L\alpha\lambda}(\nu_L)\dot{u}^\lambda \quad ; \end{aligned} \qquad (6.54)$$

$$\begin{aligned} D_{L\alpha\beta} \equiv\ & \{3\rho_L(\nu_L) - \nu_L\rho_L'(\nu_L)\}\,\frac{1}{2}\,u_{<\alpha|\beta>} \\ & + 12h_{L\alpha}(\nu_L)\dot{u}_\beta + 3h_{L\alpha}(\nu_L)_{;\beta} - 3\nu_L h_{L\alpha}'(\nu_L)\dot{u}_\beta \\ & + \frac{15}{2}\,\overset{\circ}{\pi}_{L\alpha\beta}(\nu_L) - 15\pi_{L\alpha}{}^\lambda(\nu_L)\omega_{\lambda\beta} \quad ; \end{aligned} \qquad (6.55)$$

$$D_{L\alpha\beta\gamma} \equiv \frac{15}{2}\left\{ 5\pi_{L\alpha\beta}(\nu_L)\dot{u}_\gamma - \nu_L\pi_{L\alpha\beta}'(\nu_L)\dot{u}_\gamma + \pi_{L\alpha\beta}(\nu_L)_{;\gamma}\right\}. \qquad (6.56)$$

We must equate this expression to $4\pi\nu_L^2 D_{coll}N_L$. The evaluation of $D_{coll}N_L$ in terms of the deviations from equilibrium is rather complex; hence we shall defer this calculation to chapter VII, which is devoted to such calculations. We extract the result of that calculation from equation (7.109) and equate it with equation (6.52). We obtain

$$(D_L - \tilde{D}_L) + (D_L^{\alpha} - \tilde{D}_L^{\alpha})k_{L\alpha} + (D_L^{\alpha\beta} - \tilde{D}_L^{\alpha\beta})k_{L\alpha}k_{L\beta} + D_L^{\alpha\beta\gamma}k_{L\alpha}k_{L\beta}k_{L\gamma} = 0. \qquad (6.57)$$

To extract the restrictions on the deviations from equilibrium, we multiply equation (6.57) by 1, k_L^{α}, $k_L^{\alpha}k_L^{\beta}$, $k_L^{\alpha}k_L^{\beta}k_L^{\gamma}$, and integrate over $d\Omega_L$. We deduce that the four restrictions obtained by this process are given by

$$D_L + \frac{1}{3}\Delta^{\alpha\beta}D_{L\alpha\beta} = \tilde{D}_L \quad ; \qquad (6.58)$$

$$\Delta^{\mu\lambda}D_{L\lambda} + \frac{1}{5}D_{L\alpha\beta\gamma}\Delta^{(\alpha\beta}\Delta^{\gamma\mu)} = \tilde{D}^{\mu} \quad ; \qquad (6.59)$$

$$D_{L<\alpha\beta>} = \tilde{D}_{L<\alpha\beta>} \quad ; \qquad (6.60)$$

$$D_{L\alpha}\Delta^{(\alpha\mu}\Delta^{\nu\lambda)} + \frac{1}{7}D_{L\alpha\beta\gamma}\Delta^{(\alpha\beta}\Delta^{\gamma\mu}\Delta^{\nu\lambda)} = 0 \quad . \qquad (6.61)$$

When we substitute in the appropriate expressions, from equations (6.53) to (6.56) and from (7.110) to (7.112), the first three of these results become

$$\begin{aligned} &\overset{\circ}{\rho}_L(\nu_L) + \frac{1}{3}[3\rho_L(\nu_L) - \nu_L\rho_L'(\nu_L)]\theta + h_L^{\lambda}(\nu_L)_{;\lambda} - \nu_L h_{L\lambda}'(\nu_L)\dot{u}^{\lambda} \\ &\quad = -\left\{M_{L0} + \kappa_{L1}\delta\rho_L(\nu_L) + \sum_A (M_{LA1}\delta n_A + M_{LA2}\delta\rho_A + M_{LA3}\pi_A)\right\} \; ; \end{aligned} \qquad (6.62)$$

$$\frac{1}{3}\overset{\circ}{\nabla}_{\alpha}\rho_L(\nu_L)+\frac{1}{3}[3\rho_L(\nu_L)-\nu_L\rho_L'(\nu_L)]\dot{u}_{\alpha}+\Delta_{\alpha\lambda}\overset{\circ}{h}{}^{\lambda}_L(\nu_L)$$
$$-h^{\lambda}_L(\nu_L)\omega_{\lambda\alpha}+\pi_{L\alpha}{}^{\lambda}(\nu_L)_{;\lambda}-\pi_{L\alpha}{}^{\lambda}(\nu_L)\dot{u}_{\lambda}-\nu_L\pi'_{L\alpha}{}^{\lambda}(\nu_L)\dot{u}_{\lambda} \tag{6.63}$$
$$=-\kappa_{L2}h_{L\alpha}(\nu_L)-\sum_A(M_{LA4}h_{A\alpha}+M_{LA5}q_{A\alpha}) \;;$$

$$[3\rho_L(\nu_L)-\nu_L\rho_L'(\nu_L)]\frac{1}{15}u_{<\alpha|\beta>}+\frac{1}{5}h_{L<\alpha}(\nu_L)_{;\beta}-\frac{4}{5}h_{L<\alpha}(\nu_L)\dot{u}_{\beta>}$$
$$-\frac{1}{5}\nu_L h'_{L<\alpha}(\nu_L)\dot{u}_{\beta>}+\frac{1}{2}\overset{\circ}{\pi}_{L<\alpha\beta>}(\nu_L)-\pi_{L<\alpha}{}^{\lambda}(\nu_L)\omega_{\lambda\beta>} \tag{6.64}$$
$$=-\kappa_{L3}\pi_{L\alpha\beta}(\nu_L)-\sum_A M_{LA6}\pi_{A\alpha\beta} \;.$$

The fourth result may be rearranged to produce

$$2\left\{5\pi_{L\alpha\beta}(\nu_L)\dot{u}_{\lambda}-\nu_L\pi'_{L\alpha\beta}\dot{u}_{\lambda}\right\}+\pi_{L<\alpha\beta>}(\nu_L)_{;\gamma}\Delta^{\gamma}{}_{\lambda}$$
$$+2\left\{5\pi_{L\alpha\lambda}(\nu_L)\dot{u}_{\beta}-\nu_L\pi'_{L\alpha\lambda}\dot{u}_{\beta}\right\}+\pi_{L<\alpha\lambda>}(\nu_L)_{;\gamma}\Delta^{\gamma}{}_{\beta} \tag{6.65}$$
$$+2\left\{5\pi_{L\beta\lambda}(\nu_L)\dot{u}_{\alpha}-\nu_L\pi'_{L\beta\lambda}\dot{u}_{\alpha}\right\}+\pi_{L<\beta\lambda>}(\nu_L)_{;\gamma}\Delta^{\gamma}{}_{\alpha}=0 \;.$$

We may recover the stationary case, with $\omega_{\alpha\beta}=0$, from equations (6.62) to (6.64) by neglecting the transient terms. Noting that

$$3I_{L20}(\nu_L)-\nu_L I'_{L20}(\nu_L)=\beta_L J_{L30}(\nu_L) \;, \tag{6.66}$$

we obtain for the stationary limit

$$[J_{L30}(\nu_L)/(kT_L^2)]\,\dot{T}_L+[J_{L30}(\nu_L)/3kT_L]\theta$$
$$=-M_{L0}-\kappa_{L1}\delta\rho_L(\nu_L)-\sum_A(M_{LA1}\delta n_A+M_{LA2}\delta\rho_A+M_{LA3}\pi_A) \;; \tag{6.67}$$

$$\frac{1}{3}[J_{L30}(\nu_L)/kT_L^2](\nabla_{\alpha}T_L+T_L\dot{u}_{\alpha})$$
$$=-\kappa_{L2}h_{L\alpha}(\nu_L)-\sum_A(M_{LA4}h_{A\alpha}+M_{LA5}q_{A\alpha}) \;; \tag{6.68}$$

$$\frac{1}{3}[J_{L30}(\nu_L)/(15kT_L)]u_{<\alpha|\beta>} = -\kappa_{L3}\pi_{L\alpha\beta}(\nu_L) - \sum_A M_{LA6}\pi_{A\alpha\beta} \quad . \qquad (6.69)$$

If we choose $T_L = T_M$ and neglect the deviations from equilibrium of the massive particles, then we obtain, for the case of photons, the results reported by Straumann [35].

E. Frequency Averaging

To integrate the transport equations over frequency we must define some way to average κ_{Li} and M_{LAi} over frequency. In the stationary case for photons where the contributions by massive particles is negligible, we would just divide equations (6.67) to (6.69) by the appropriate κ_{Li} and integrate. The appropriate average in that case is the Rosseland mean [3,35]:

$$\frac{1}{\bar{\kappa}_{Li}} \equiv \frac{\int \frac{1}{\kappa_{Li}} \frac{\partial I_{L20}(\nu_L, T_L)}{\partial T_L} d\nu_L}{\int \frac{\partial I_{L20}(\nu_L, T_L)}{\partial T_L} d\nu_L} = \frac{\frac{1}{3}\int \frac{1}{\kappa_{Li}} J_{L30}(\nu_L) d\nu_L}{\frac{1}{3}\int J_{L30}(\nu_L) d\nu_L} \quad . \qquad (6.70)$$

We shall carry this averaging technique over to the transient case.

Strictly speaking $\bar{\kappa}_{L2}$ and $\bar{\kappa}_{L3}$ should be tensors instead of scalars. However, this is inconvenient and probably not much more accurate than if they are scalars (Anderson and Spiegel [2]). Also, if these tensor absorption coefficients are spatially isotropic, they will reduce essentially to a scalar coefficient. For these reasons we shall take all of our absorption coefficients to be scalars.

Let us define the following frequency averages:

$$\bar{M}_{LAi} \equiv \int \frac{M_{LAi}}{\kappa_{L1}}\, d\nu_L \;, \; i = 0, 1, 2, 3; \tag{6.71}$$

$$\bar{M}_{LAi} \equiv \int \frac{M_{LAi}}{\kappa_{L2}}\, d\nu_L \;, \; i = 4, 5; \tag{6.72}$$

$$\bar{M}_{LA6} \equiv \int \frac{M_{LA6}}{\kappa_{L3}}\, d\nu_L \;. \tag{6.73}$$

Then, integration over frequency of equations (6.62) to (6.65) produces

$$\begin{aligned} &\dot{\rho}_L + \frac{4}{3}\rho_L\theta + h_L{}^{\lambda}{}_{|\lambda} + h_L^{\lambda}\dot{u}_\lambda \\ &\quad = -\bar{\kappa}_{L1}\left\{ \bar{M}_{L0} + \delta\rho_L + \sum_A (\bar{M}_{LA1}\delta n_A + \bar{M}_{LA2}\delta\rho_A + \bar{M}_{LA3}\pi_A) \right\} \;; \end{aligned} \tag{6.74}$$

$$\begin{aligned} &\frac{1}{3}\nabla_\alpha\rho_L + \frac{4}{3}\rho_L\dot{u}_\alpha + \Delta_\alpha{}^{\lambda}h_{L\lambda} - h_L^{\lambda}\omega_{\lambda\alpha} + \pi_{L\alpha}{}^{\lambda}{}_{|\lambda} \\ &\quad = -\bar{\kappa}_{L2}\left\{ h_{L\alpha} + \sum_A (\bar{M}_{LA4}h_{A\alpha} + \bar{M}_{LA5}q_{A\alpha}) \right\} \;; \end{aligned} \tag{6.75}$$

$$\frac{4\rho_L}{15} u_{<\alpha|\beta>} + \frac{1}{5} h_{L<\alpha|\beta>} + h_{L<\alpha}\dot{u}_{\beta>} + \frac{1}{2} \dot{\pi}_{L<\alpha\beta>} - \pi^{\lambda}_{L<\alpha}\omega_{\lambda\beta>}$$
$$= - \bar{\kappa}_{L3} \left\{ \pi_{L\alpha\beta} + \sum_A \bar{M}_{LA6}\pi_{A\alpha\beta} \right\} \quad ; \tag{6.76}$$

$$12 \left\{ \pi_{L<\alpha\beta>}\dot{u}_{\lambda} + \pi_{L<\alpha\lambda>}\dot{u}_{\beta} + \pi_{L<\beta\lambda>}\dot{u}_{\alpha} \right\}$$
$$+ \pi_{L<\alpha\beta>|\gamma}\Delta^{\gamma}{}_{\lambda} + \pi_{L<\alpha\lambda>|\gamma}\Delta^{\gamma}{}_{\beta} + \pi_{L<\beta\lambda>|\gamma}\Delta^{\gamma}{}_{\alpha} = 0 \quad . \tag{6.77}$$

These are our transport equations. In the stationary limit, with $T_L = T_M$, and neglecting the deviations from equilibrium of the massive species, the first three of these equations are identical with the results of Thomas [37] and Anderson and Spiegel [2] recast into our notation, provided we assume that all of the absorption coefficients are the same. We also note that the first two of these equations are expressions for $-u_\alpha T^{\alpha\lambda}_{L}|_\lambda$ and $\Delta_{\mu\alpha} T^{\alpha\lambda}_{L}|_\lambda$ respectively.

F. Coefficients for Dissipative Processes

To identify the coefficients of bulk viscosity, shear viscosity, and heat flux, we must examine the entropy flux and entropy production. The entropy flux as defined by equation (2.22) becomes

$$S^{\alpha}_{L} = kI_{L21}\beta_L u^{\alpha} - k\beta_{L\lambda}T^{\lambda\alpha}_{L} - Q^{\alpha}_{L} \quad ; \tag{6.78}$$

where Q_L^α is given by

$$\begin{aligned} Q_L^\alpha \equiv \frac{k}{2} \int (\mathring{N}_L \mathring{\Delta}_L / g_L) f_L^2 w_L^\alpha dV_L &= \frac{k}{2} u^\alpha \Big\{ \int \delta\rho_L^2(\nu_L) d\nu_L \\ &+3k \int h_L^\lambda(\nu_L) h_{L\lambda}(\nu_L) d\nu_L + \frac{15k}{3} \int \pi_L^{\alpha\beta}(\nu_L) \pi_{L\alpha\beta}(\nu_L) d\nu_L \Big\} \qquad (6.79) \\ &+ k \int \delta\rho_L(\nu_L) h_L^\alpha(\nu_L) d\nu_L + 3k \int h_L^\lambda(\nu_L) \pi_{L\lambda}{}^\alpha(\nu_L) d\nu_L \quad . \end{aligned}$$

Since decomposition of $\Phi_L(N_L)$ up to second order, equation (4.95), is invariant under fitting and frame changes, equations (4.96) and (4.97), then equation (6.78) is also invariant under these transformations. In this case, we have just reproduced the massive particle result for the entropy flux, equation (4.67), except that in this case Q_L^α is given by equation (6.79); however, this latter expression can not be credibly integrated over frequency to produce an expression like (4.77).

Let us compute the entropy production. We simply differentiate and contract (7.78). This produces

$$\begin{aligned} S_{L|\alpha}^\alpha = k\delta\rho_L(\dot{\beta}_L - \frac{1}{3}\beta_L\theta) - k\beta_L u_\alpha T_L^{\alpha\lambda}{}_{|\lambda} \qquad (6.80) \\ + kh_L^\lambda(\nabla_\lambda\beta_L - \beta_L\dot{u}_\lambda) - k\beta_L\sigma_{\mu\nu}\pi_L^{\mu\nu} - Q_{L|\alpha}^\alpha \quad . \end{aligned}$$

Now equations (6.75) and (6.76) may be written as

$$\begin{aligned} h_{L\alpha} = -\frac{1}{\kappa_{L2}} \Big\{ - J_{L30}(\nabla_\alpha\beta_L - \beta_L\dot{u}_\alpha) + \frac{4}{3}\delta\rho_L\dot{u}_\alpha + \frac{1}{3}\nabla_\alpha\delta\rho_L \qquad (6.81) \\ + \Delta_{\alpha\lambda}\dot{h}_L^\lambda - h_L^\lambda\omega_{\lambda\alpha} + \pi_{L\alpha}{}^\lambda{}_{|\lambda} \Big\} - \sum_A (\bar{M}_{LA4}h_{A\alpha} + \bar{M}_{LA5}q_{A\alpha}) \quad ; \end{aligned}$$

$$\pi_{L\alpha\beta} = -\frac{1}{\bar{\kappa}_{L3}}\left\{\frac{4I_{L20}}{15}\,u_{<\alpha|\beta>} + \frac{1}{5}\,h_{L<\alpha|\beta>} + h_{L<\alpha}\dot{u}_{\beta>} + \frac{1}{2}\dot{\pi}_{L<\alpha\beta>} - \pi_{L<\alpha\lambda}\omega^{\lambda}{}_{\beta>}\right\} - \sum_A \bar{M}_{LA6}\pi_{L\alpha\beta} \quad . \tag{6.82}$$

When we substitute these results into (6.80) we obtain

$$S^{\alpha}_{L|\alpha} = k\delta\rho_L(\dot{\beta}_L - \frac{1}{3}\beta_L\theta) - k\beta_L u_\alpha T_L^{\alpha\lambda}{}_{|\lambda} + \lambda_L\,(\nabla_\alpha T_L + T_L\dot{u}_\alpha)(\nabla^\alpha T_L + T_L\dot{u}^\alpha) + \tilde{\nu}_L\,\sigma_{\mu\nu}\sigma^{\mu\nu} + Q_L \quad ; \tag{6.83}$$

where Q_L is given by

$$Q_L \equiv -Q^{\alpha}_{L|\alpha} - \frac{k}{\bar{\kappa}_{L2}}(\nabla^\alpha\beta_L - \beta_L\dot{u}^\alpha)\left\{\frac{1}{3}\nabla_\alpha\delta\rho_L + \frac{4}{3}\delta\rho_L\dot{u}_\alpha + \Delta_\alpha{}^{\lambda}\dot{h}_{L\lambda} - h_L^{\lambda}\omega_{\lambda\alpha} + \pi_{L\alpha}{}^{\lambda}{}_{|\lambda} + \bar{\kappa}_{L2}\sum_A(\bar{M}_{LA4}h_{A\alpha} + \bar{M}_{LA5}q_{A\alpha})\right. $$
$$+ \frac{k}{\bar{\kappa}_{L3}}\beta_L\sigma^{\alpha\beta}\left\{\frac{1}{5}h_{L<\alpha|\beta>} + h_{L<\alpha}\dot{u}_{\beta>} + \frac{1}{2}\dot{\pi}_{L<\alpha\beta>} - \pi_{L<\alpha\lambda}\omega^{\lambda}{}_{\beta>} + \bar{\kappa}_{L3}\sum_A\bar{M}_{LA6}\pi_{A\alpha\beta}\right\} \quad ; \tag{6.84}$$

and the coefficients of heat flux and shear viscosity are given by

$$\lambda_L \equiv J_{L30}/(k\bar{\kappa}_{L2}T_L^4) \quad ; \tag{6.85}$$

$$\tilde{\nu}_L \equiv 8k\beta_L I_{L20}/(15\bar{\kappa}_{L3}) \quad . \tag{6.86}$$

To determine the bulk viscosity we must examine the entropy production of the whole gas. We shall assume for simplicity that the deviations from equilibrium for massive particles are negligible. We shall also take $T_L = T_M$. In this case, equation (5.54) for species A becomes

$$S^{\alpha}_{A|\alpha} = -k\tilde{\beta}^{\lambda}_{M} T_{A\lambda}{}^{\alpha}{}_{|\alpha} \quad . \tag{6.87}$$

When we sum this result over all massive species we obtain the entropy production for the matter:

$$S^{\alpha}_{M|\alpha} = -k\tilde{\beta}^{\lambda}_{M} T_{M\lambda}{}^{\alpha}{}_{|\alpha} \quad . \tag{6.88}$$

Since the total energy-momentum tensor of the gas is conserved we must have that

$$T^{\alpha\lambda}_{M}{}_{|\lambda} = -\sum_{L} T^{\alpha\lambda}_{L}{}_{|\lambda} \quad . \tag{6.89}$$

Hence, summing equation (6.83) over all massless species and adding it to equation (6.88) gives us the total entropy production of the gas:

$$\begin{aligned} S^{\alpha}{}_{|\alpha} = {} & \sum_{L} k\delta\rho_{L}(\dot{\tilde{\beta}}_{M} - \frac{1}{3}\tilde{\beta}_{M}\theta) + \sum_{L}\lambda_{L}(\nabla^{\alpha}T_{M} + T_{M}\dot{u}^{\alpha})(\nabla_{\alpha}T_{M} + T_{M}\dot{u}_{\alpha}) \\ & + \sum_{L}\tilde{\nu}_{L}\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \sum_{L} Q_{L} \quad . \end{aligned} \tag{6.90}$$

We may solve for $\delta\rho_L$ with $T_L = T_M$ via equation (6.74). We have that

$$\delta\rho_{L} = + \frac{J_{L30}}{\bar{\kappa}_{L1}}(\dot{\tilde{\beta}}_{M} - \frac{1}{3}\tilde{\beta}_{M}\theta) - \frac{h^{\alpha}_{L|\alpha}}{\bar{\kappa}_{L1}} - \frac{h^{\alpha}_{L}\dot{u}_{\alpha}}{\bar{\kappa}_{L1}} - \frac{\delta\dot{\rho}_{L}}{\bar{\kappa}_{L1}} \quad . \tag{6.91}$$

since $\bar{M}_{L0}=0$ for a common temperature. When we substitute this expression into equation (6.90) and define

$$\tilde{Q}_L \equiv Q_L - (\bar{\kappa}_{L1})^{-1}(\delta\dot{\rho}_L + h^{\alpha}_{L|\alpha} + h^{\alpha}_L\dot{u}_{\alpha})(\dot{\tilde{\beta}}_M - \frac{1}{3}\tilde{\beta}_M\theta) \quad , \tag{6.92}$$

we obtain

$$\begin{aligned} S^{\alpha}{}_{|\alpha} = k\sum_L(J_{L30}/\bar{\kappa}_{L1})(\dot{\tilde{\beta}}_M - \frac{1}{3}\tilde{\beta}_M\theta)^2 + \sum_L \tilde{\nu}_L\sigma_{\alpha\beta}\sigma^{\alpha\beta} \\ + \sum_L \lambda_L(\nabla^{\alpha}T_M + T_M\dot{u}^{\alpha})(\nabla_{\alpha}T_M + T_M\dot{u}_{\alpha}) + \sum_L \tilde{Q}_L \quad . \end{aligned} \tag{6.93}$$

Let us now obtain an expression for $\dot{\tilde{\beta}}_M$. From equation (5.63) for $\dot{\beta}_A$ we obtain for species A the following result with $\tilde{\beta}_A=\tilde{\beta}_M$:

$$m_A\frac{D_{A20}}{J_{A10}}\dot{\tilde{\beta}}_M = -\tilde{\beta}_M m_A[(J_{A20}J_{A21}-J_{A10}J_{A31})/J_{A10}]\theta + u_{\alpha}T^{\alpha\lambda}_{A|\lambda} . \tag{6.94}$$

Summing this result over all massive species A and employing (6.89) we obtain

$$\begin{aligned} \sum_A m_A\frac{D_{A20}}{J_{A10}}\dot{\tilde{\beta}}_M = -\tilde{\beta}_M\sum_A m_A[(J_{A20}J_{A21}-J_{A10}J_{A31})/J_{A10}]\theta \\ - u_{\alpha}\sum_L T^{\alpha\lambda}_L{}_{|\lambda} \quad . \end{aligned} \tag{6.95}$$

When we substitute for $-u_{\alpha}T^{\alpha\lambda}_{A}{}_{|\lambda}$, which is given by the left hand side of equation (6.74), we obtain

$$\begin{aligned} \left\{\sum_A m_A\frac{D_{A20}}{J_{A10}} + \sum_L J_{L30}\right\}\dot{\tilde{\beta}}_M = \sum_L(\delta\dot{\rho}_L + h^{\alpha}_{L|\alpha} + h^{\lambda}_L\dot{u}_{\lambda}) \\ + \tilde{\beta}_M\left\{\sum_A m_A[(J_{A10}J_{A31}-J_{A20}J_{A21})/J_{A10}] + \sum_L J_{L31}\right\}\theta \quad . \end{aligned} \tag{6.96}$$

This expression may be more conveniently expressed. When we hold the individual number densities constant, i.e. $dn_A = 0$, we obtain for any thermodynamic change that

$$d\alpha_A = m_A (J_{A20}/J_{A10})\, d\tilde{\beta}_M \quad . \tag{6.97}$$

Applying this condition to the energy density gives us

$$\left(\frac{\partial\rho_A}{\partial\tilde{\beta}_M}\right)_{n_A} = -\, m_A (D_{A20}/J_{A10}) \quad . \tag{6.98}$$

Since the actual thermodynamical pressure of species A is I_{A21} to order zero, we also have, via (6.97), that

$$\left(\frac{\partial P_A}{\partial\tilde{\beta}_M}\right)_{n_A} = -\, m_A [(J_{A10}J_{A31} - J_{A20}J_{A21})/J_{A10}] \quad . \tag{6.99}$$

Similarly, for the massless species we obtain

$$\left(\frac{\partial\rho_L}{\partial\tilde{\beta}_M}\right)_{n_A} = -\, J_{L30} \quad ; \left(\frac{\partial P_L}{\partial\tilde{\beta}_M}\right)_{n_A} = -\, J_{L31} \quad . \tag{6.100}$$

Let us now sum these expressions over all species in the gas. This provides us with two convenient results:

$$\left(\frac{\partial\rho}{\partial\tilde{\beta}_M}\right)_{n_A} = -\left\{ \sum_A m_A (D_{A20}/J_{A10}) + \sum_L J_{L30} \right\} \quad ; \tag{6.101}$$

$$\left(\frac{\partial P}{\partial\tilde{\beta}_M}\right)_{n_A} = -\left\{ \sum_A m_A [(J_{A10}J_{A31} - J_{A20}J_{A21})/J_{A10}] + \sum_L J_{L31} \right\} . \tag{6.102}$$

We may now substitute these two results into expression (6.96); therefore, we obtain the following result:

$$\dot{\tilde{\beta}}_M = \tilde{\beta}_M \theta (\partial P/\partial \rho)_{n_A} - \left\{ \sum_L (\delta \dot{\rho}_L + h^\lambda_L{}_{|\lambda} + h^\lambda_L \dot{u}_\lambda) \right\} \div (\partial \rho / \partial \tilde{\beta}_M)_{n_A} \quad . \qquad (6.103)$$

Consequently, we conclude that

$$\begin{aligned} \dot{\tilde{\beta}}_M - \frac{1}{3} \tilde{\beta}_M \theta = & - \tilde{\beta}_M \left\{ \frac{1}{3} - (\partial P/\partial \rho)_{n_A} \right\} \theta \\ & - \left\{ \sum_L (\delta \dot{\rho}_L + h^\lambda_L{}_{|\lambda} + h^\lambda_L \dot{u}_\lambda \right\} \div (\partial \rho / \partial \tilde{\beta}_M)_{n_A} \quad . \end{aligned} \qquad (6.104)$$

We may now obtain the coefficient of bulk viscosity. We substitute equation (6.104) into (6.93) and obtain

$$\begin{aligned} S^\alpha{}_{|\alpha} = & \sum_L \tilde{\nu}_L \sigma^{\alpha\beta} \sigma_{\alpha\beta} + \nu_b \theta^2 \\ & + \sum_L \lambda_L (\nabla^\alpha T_M + T_M \dot{u}^\alpha)(\nabla_\alpha T_M + T_M \dot{u}_\alpha) + \bar{Q} \quad ; \end{aligned} \qquad (6.105)$$

where we have defined

$$\begin{aligned} \bar{Q} \equiv & \left\{ \sum_L \frac{J_{L30}}{\bar{\kappa}_{L1}} \right\} 2 \tilde{\beta}_M \left[\frac{1}{3} - \left(\frac{\partial P}{\partial \rho} \right)_{n_A} \right] \left\{ \sum_L (\delta \dot{\rho}_L + h^\lambda_L{}_{|\lambda} + h^\lambda_L \dot{u}_\lambda) \right\} \theta \div (\partial \rho / \partial \tilde{\beta}_M)_{n_A} \\ & + \left\{ \sum_L \frac{J_{L30}}{\bar{\kappa}_{L1}} \right\} \times \left\{ \sum_L (\delta \dot{\rho}_L + h^\lambda_L{}_{|\lambda} + h^\lambda_L \dot{u}_\lambda \div (\partial \rho / \tilde{\beta}_M)_{n_A} \right\}^2 + \sum_L \tilde{Q}_L \quad ; \end{aligned} \qquad (6.106)$$

and the coefficient of bulk viscosity is given by

$$\nu_b \equiv \tilde{\beta}_M \left\{ \sum_L J_{L30} / \bar{\kappa}_{L1} \right\} \left\{ \frac{1}{3} - (\partial P/\partial \rho)_{n_A} \right\}^2 \quad . \qquad (6.107)$$

We note that in the stationary limit and for only one massless component in the gas, we have precisely the result for the coefficient of bulk viscosity first derived by

Weinberg [41]. Hence our result is the generalization to the case where more than one massless species is in the gas. We note that if the matter is itself highly relativistic then the matter pressure is about one-third the matter density, i.e. $P \sim \rho/3$; in this case the bulk viscosity vanishes.

We noted previously that, for the sake of consistency, we must include the contribution of the interactions between massive and massless particles in $D_{coll}N_A$. Consequently, such contributions must show up in the transport equations for massive particles. This contribution is computed in chapter VII and leads to equations (7.78) and (7.79). These contributions have exactly the same structure as the contributions due to massive particles, equations (7.41) and (7.42). Thus, all we have to do is incorporate equations (7.70) and (7.71) into (7.41) and (7.42) by redefining the L_{ABi} and χ_{ABi} by

$$L_{ABi} \leftarrow L_{ABi} + L'_{ABi} \; ; \tag{6.108}$$

$$\chi'_{ABi} \leftarrow \chi_{ABi} + \chi'_{ABi} \; . \tag{6.109}$$

Then the transport equations for massive particles retain their structure as reported in chapter V except that now they include interactions with massless particles.

This concludes our analysis of the massless case. We must now perform those computations of the collisional structures that we deferred in this chapter and in chapter V; these computations are performed in chapter VII.

VII. Non-equilibrium Conservation Laws

In chapters V and VI we required a knowledge of the structures of $T_{A\ |\lambda}^{\alpha\lambda}$ and $U_{A\ |\lambda}^{\alpha\beta\lambda}$ in terms of the collision probabilities and the deviations from equilibrium. These structures were employed to write out the final forms of the transport equations. In this chapter it will be our task to deduce what those structures are.

Our calculations in this chapter are of a highly technical nature; thus we shall present a brief overview of how our calculations done.

To perform our calculations we first specify some assumptions and perform some preliminary calculations to facilitate subsequent calculations. Then we shall split up our discussion into three stages. The first stage consists of examining the collision structures of $T_{A|\lambda}^{\alpha\lambda}$ and $U_{A\ |\lambda}^{\alpha\beta\lambda}$ for massive particles due to interactions between massive particles alone. The second stage consists of calculating the additional terms necessary to include the interactions between massive and massless particles. The third stage consists of determining the structure of $D_{\text{coll}}N_L$ for massless particles due to interactions between massless particles and massive particles, considering interactions with other massless particles to be negligible.

In each of these stages cited above we will proceed by expressing the distribution function in the collision integrals via the appropriate relativistic Grad method. The resulting expressions in terms of the equilibrium distribution function and the deviation from equilibrium, f_A, permits us to define several classes of integrals to simplify the expressions. We analyse these integrals in terms of their irreducible structures. Finally, these irreducible structures are introduced into the collision integrals to obtain the final expressions required to obtain the transport equations in the previous chapters.

A. Assumptions and Preliminary Calculations

Let us begin by assuming that we have detailed balancing for all binary collisions, creation processes, and absorption processes:

$$W(p_A,p_B|\overset{*}{p}_A,\overset{*}{p}_B) = W(\overset{*}{p}_A,\overset{*}{p}_B|p_A,p_B) \quad ; \tag{7.1}$$

$$W(p_A,p_B,p_L|\overset{*}{p}_A,\overset{*}{p}_B) = W(\overset{*}{p}_A,\overset{*}{p}_B|p_A,p_B,p_L) \quad .$$

This assumption is justified by the microscopic reversibility of these processes (Anderson [1], Israel [12]). For convenience we shall denote the transition probablities by W_{AB} and W_{ABL} . In equilibrium, the assumption of detailed balancing and the functional form of the distribution funcion $\mathring{N}_A$ implies that $D_{coll}N_A=0$ for all species. Consequently, $T^{\alpha\lambda}_{A|\lambda}$ and $U^{\alpha\beta\lambda}_{A\ |\lambda}$ are identically zero in equilibrium. Therefore in non-equilibrium we should expect these quantities to be of order one; furthermore, we expect them to be expressible in terms of the deviations from equilibrium $\delta\rho_A$, h^{α}_A , etc.

Before we enter into the main discussion, let us derive some preliminary results. When we employ the most general fitting conditions, each species has its own thermal potential α_A and inverse temperature $\tilde{\beta}_A$. However, each is close to a common $\tilde{\beta}$ to first order, that is, we may write $\tilde{\beta}_A = \tilde{\beta}+\delta\tilde{\beta}_A$ where $\delta\tilde{\beta}_A$ is order one. Therefore we conclude, after some algebra, that for binary collisions

$$\exp\left\{\beta^{\lambda}_{A}(w_{A\lambda}-\overset{*}{w}_{A\lambda})+\beta^{\lambda}_{B}(w_{B\lambda}-\overset{*}{w}_{B\lambda})\right\} = 1+\beta_{A\lambda}(w^{\lambda}_{A}-\overset{*}{w}{}^{\lambda}_{A})+\beta_{B\lambda}(w^{\lambda}_{B}-\overset{*}{w}{}^{\lambda}_{B}); \quad (7.2)$$

where we have employed the relation $\tilde{\beta}^{\lambda}[p_{\lambda}]=0$ which implies that the quantity which appears in the exponent in equation (7.2) is order one. Since $[\alpha]=0$ equation (7.2) implies that

$$\overset{\circ *}{N}_{A}\overset{\circ *}{N}_{B}\overset{\circ}{\Delta}_{A}\overset{\circ}{\Delta}_{B} = \overset{\circ}{N}_{A}\overset{\circ}{N}_{B}\overset{\circ *}{\Delta}_{A}\overset{\circ *}{\Delta}_{B}\{1 + \beta_{A\lambda}(\overset{*}{w}{}^{\lambda}_{A}-w^{\lambda}_{A}) + \beta_{B\lambda}(\overset{*}{w}{}^{\lambda}_{B}-w^{\lambda}_{B})\} \quad , \qquad (7.3)$$

for all species. A similar analysis for absorption and creation processes leads us to conclude that

$$\overset{\circ *}{N}_{A}\overset{\circ *}{N}_{B}\overset{\circ}{\Delta}_{A}\overset{\circ}{\Delta}_{B}\overset{\circ}{\Delta}_{L} = \overset{\circ}{N}_{A}\overset{\circ}{N}_{B}\overset{\circ}{N}_{L}\overset{\circ *}{\Delta}_{A}\overset{\circ *}{\Delta}_{B}\{1+\beta_{A\lambda}(\overset{*}{w}{}^{\lambda}_{A}-w^{\lambda}_{A})+\beta_{B\lambda}(\overset{*}{w}{}^{\lambda}_{B}-w^{\lambda}_{B})-\beta_{L\lambda}w^{\lambda}_{L}\}, \quad (7.4)$$

$$\overset{\circ *}{N}_{A}\overset{\circ *}{N}_{B}\overset{\circ}{N}_{L}\overset{\circ}{\Delta}_{A}\overset{\circ}{\Delta}_{B} = \overset{\circ}{N}_{A}\overset{\circ}{N}_{B}\overset{\circ}{\Delta}_{L}\overset{\circ *}{\Delta}_{A}\overset{\circ *}{\Delta}_{B}\{1+\beta_{A\lambda}(\overset{*}{w}{}^{\lambda}_{A}-w^{\lambda}_{A})+\beta_{B\lambda}(\overset{*}{w}{}^{\lambda}_{B}-w^{\lambda}_{B})+\beta_{L\lambda}w^{\lambda}_{L}\}, \quad (7.5)$$

for absorption and creation processes respectively.

Consider the decomposition of N_A in terms of $\overset{\circ}{N}_A$ and f_A via equations (4.10) and (4.12). Employing these structures allows us to conclude for binary collisions that

$$\begin{aligned} N_A N_B \overset{*}{\Delta}_A \overset{*}{\Delta}_B = \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B \{ 1 &+ \overset{\circ}{\Delta}_A f_A/g_A \\ &+ \overset{\circ}{\Delta}_B f_B/g_B + \varepsilon_A \overset{\circ *}{N}_A \overset{*}{f}_A/g_A + \varepsilon_B \overset{\circ *}{N}_B \overset{*}{f}_B/g_B \} \ . \end{aligned} \tag{7.6}$$

Interchanging starred and unstarred variables, and using (7.3) we obtain

$$\begin{aligned} \overset{*}{N}_A \overset{*}{N}_B \Delta_A \Delta_B = \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B \{1 &+ \overset{\circ *}{\Delta}_A \overset{*}{f}_A/g_A + \overset{\circ *}{\Delta}_B \overset{*}{f}_B/g_B \\ &+ \varepsilon_A \overset{\circ}{N}_A f_A/g_A + \varepsilon_B \overset{\circ}{N}_B f_B/g_B + \beta_{A\lambda}(\overset{*}{w}{}^{\lambda}_A - w^{\lambda}_A) + \beta_{B\lambda}(\overset{*}{w}{}^{\lambda}_B - w^{\lambda}_B)\} \ . \end{aligned} \tag{7.7}$$

A similar analysis for absorption processes in which we employ equation (7.4) gives us

$$\begin{aligned} N_A N_B N_L \overset{*}{\Delta}_A \overset{*}{\Delta}_B = \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B \{1 &+ \overset{\circ}{\Delta}_A f_A/g_A + \overset{\circ}{\Delta}_B f_B/g_B \\ &+ \overset{\circ}{\Delta}_L f_L/g_L + \varepsilon_A \overset{\circ *}{N}_A \overset{*}{f}_A/g_A + \varepsilon_B \overset{\circ *}{N}_B \overset{*}{f}_B/g_B\} \quad ; \end{aligned} \tag{7.8}$$

$$\begin{aligned} \overset{*}{N}_A \overset{*}{N}_B \Delta_L \Delta_A \Delta_B = \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B \{1 &+ \overset{\circ *}{\Delta}_A \overset{*}{f}_A/g_A + \overset{\circ *}{\Delta}_B \overset{*}{f}_B/g_B \\ &+ \varepsilon_L \overset{\circ}{N}_L f_L/g_L + \varepsilon_A \overset{\circ}{N}_A f_A/g_A + \varepsilon_B \overset{\circ}{N}_B f_B/g_B \\ &+ \beta_{A\lambda}(\overset{*}{w}{}^{\lambda}_A - w^{\lambda}_A) + \beta_{B\lambda}(\overset{*}{w}{}^{\lambda}_B - w^{\lambda}_B) - \beta_{L\lambda} w^{\lambda}_L\} \quad . \end{aligned} \tag{7.9}$$

Also, for creation processes, for which we employ equation (7.5), we obtain

$$\begin{aligned} N_A N_B \Delta_L \overset{*}{\Delta}_A \overset{*}{\Delta}_B = \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{\Delta}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B \{1 &+ \overset{\circ}{\Delta}_A f_A/g_A + \overset{\circ}{\Delta}_B f_B/g_B \\ &+ \varepsilon_L \overset{\circ}{N}_L f_L/g_L + \varepsilon_A \overset{\circ *}{N}_A \overset{*}{f}_A/g_A + \varepsilon_B \overset{\circ *}{N}_B \overset{*}{f}_B/g_B\} \ ; \end{aligned} \tag{7.10}$$

$$\begin{aligned} \overset{*}{N}_A \overset{*}{N}_B N_L \Delta_A \Delta_B = \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{\Delta}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B \{1 &+ \overset{\circ *}{\Delta}_A \overset{*}{f}_A/g_A + \overset{\circ *}{\Delta}_B \overset{*}{f}_B/g_B \\ &+ \overset{\circ}{\Delta}_L f_L/g_L + \varepsilon_A \overset{\circ}{N}_A f_A/g_A + \varepsilon_B \overset{\circ}{N}_B f_B/g_B \\ &+ \beta_{A\lambda}(\overset{*}{w}{}^{\lambda}_A - w^{\lambda}_A) + \beta_{B\lambda}(\overset{*}{w}{}^{\lambda}_B - w^{\lambda}_B) + \beta_{L\lambda} w^{\lambda}_L \} \quad . \end{aligned} \tag{7.11}$$

Let us now subtract equation (7.6) from (7.7), (7.8) from (7.9), and (7.10) from (7.11). We obtain for binary collisions, absorptions and creations respectively

$$\begin{aligned}&\overset{*}{N}_A\overset{*}{N}_B\Delta_A\Delta_B - N_A N_B\overset{*}{\Delta}_A\overset{*}{\Delta}_B \\ &\quad = \overset{\circ}{N}_A\overset{\circ}{N}_B\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_B\{\overset{*}{f}_A+\overset{*}{f}_B-f_A-f_B + \beta_{A\lambda}(\overset{*}{w}{}^{\lambda}_A-w^{\lambda}_A)+\beta_{B\lambda}(\overset{*}{w}{}^{\lambda}_B-w^{\lambda}_B)\}\ ;\end{aligned} \tag{7.12}$$

$$\begin{aligned}&\overset{*}{N}_A\overset{*}{N}_B\Delta_L\Delta_A\Delta_B - N_A N_B N_L\Delta_A\Delta_B = \overset{\circ}{N}_A\overset{\circ}{N}_B\overset{\circ}{N}_L\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_B\{\overset{*}{f}_A+\overset{*}{f}_B-f_A-f_B-f_L \\ &\quad + \beta_{A\lambda}(\overset{*}{w}{}^{\lambda}_A-w^{\lambda}_A) + \beta_{B\lambda}(\overset{*}{w}{}^{\lambda}_B-w^{\lambda}_B) - \beta^{\lambda}_L w_{L\lambda}\}\ ;\end{aligned} \tag{7.13}$$

$$\begin{aligned}&\overset{*}{N}_A\overset{*}{N}_B N_L\Delta_A\Delta_B - N_A N_B\Delta_L\overset{*}{\Delta}_A\overset{*}{\Delta}_B = \overset{\circ}{N}_A\overset{\circ}{N}_B\overset{\circ}{\Delta}_L\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_B\{f_A+f_B+f_L-f_A-f_B \\ &\quad + \beta_{A\lambda}(\overset{*}{w}{}^{\lambda}_A-w^{\lambda}_A) + \beta_{B\lambda}(\overset{*}{w}{}^{\lambda}_B-w^{\lambda}_B) + \beta_{L\lambda}w^{\lambda}_L\}\ .\end{aligned} \tag{7.14}$$

We may express the Boltzmann collision terms for massive and massless species via equations (7.12), (7.13), and (7.14). Let us reserve the subscripts A, B for timelike particles and the subscript L to refer to massless particles only. Thus we have for massive particles

$$\begin{aligned}D_{coll}N_A &= \sum_B \int \overset{\circ}{N}_A\overset{\circ}{N}_B\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_B(\overset{*}{f}_A+\overset{*}{f}_B-f_A-f_B)W_{AB}\,dV_B\,d\overset{*}{V}_A\,d\overset{*}{V}_B \\ &+ \sum_B\int\overset{\circ}{N}_A\overset{\circ}{N}_B\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_B\{\beta_{A\lambda}(\overset{*}{w}{}^{\lambda}_A-w^{\lambda}_A)+\beta_{B\lambda}(\overset{*}{w}{}^{\lambda}_B-w^{\lambda}_B)\}W_{AB}\,dV_B\,d\overset{*}{V}_A\,d\overset{*}{V}_B \\ &+ \tilde{D}_{coll}N_A\ ;\end{aligned} \tag{7.15}$$

where $\tilde{D}_{coll}N_A$, the contribution to $D_{coll}N_A$ from interactions with massless particles, is given by

$$\begin{aligned}
\tilde{D}_{\text{coll}}N_A = {} & \sum_L \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L (\overset{*}{f}_A + \overset{*}{f}_L - f_A - f_L) W_{AL}\, d\overset{*}{V}_A\, dV_L\, d\overset{*}{V}_L \\
& + \sum_L \sum_B \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{f}_A + \overset{*}{f}_B - f_A - f_B - f_L) W_{ABL}\, dV_B\, d\overset{*}{V}_A\, d\overset{*}{V}_B\, dV_L \\
& + \sum_L \sum_B \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{\Delta}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{f}_A + \overset{*}{f}_B + f_L - f_A - f_B) W_{ABL}\, dV_B\, d\overset{*}{V}_A\, d\overset{*}{V}_B\, dV_L \\
& + \sum_L \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L \{\beta_{A\lambda} (\overset{*\lambda}{w}_A - w_A^{\lambda}) + \beta_{L\lambda} (\overset{*\lambda}{w}_L - w_L^{\lambda})\} W_{AL}\, d\overset{*}{V}_A\, dV_L\, d\overset{*}{V}_L \\
& + \sum_L \sum_B \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B \{\beta_{A\lambda} (\overset{*\lambda}{w}_A - w_B^{\lambda}) + \beta_{B\lambda} (\overset{*\lambda}{w}_B - w_B^{\lambda}) - \beta_{L\lambda} w_L^{\lambda}\} W_{ABL}\, dV_B\, d\overset{*}{V}_A\, d\overset{*}{V}_B\, dV_L \\
& + \sum_L \sum_B \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{\Delta}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B \{\beta_{A\lambda} (\overset{*\lambda}{w}_A - w_A^{\lambda}) + \beta_{B\lambda} (\overset{*\lambda}{w}_B - w_B^{\lambda}) + \beta_{L\lambda} w_L^{\lambda}\} W_{ABL}\, dV_B\, d\overset{*}{V}_A\, d\overset{*}{V}_B\, dV_L .
\end{aligned} \tag{7.16}$$

For massless particles we have that

$$\begin{aligned}
D_{\text{coll}}N_L = {} & \sum_A \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L (\overset{*}{f}_A + \overset{*}{f}_L - f_A - f_L) W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L \\
& + \sum_A \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L \{\beta_{A\lambda} (\overset{*\lambda}{w}_A - w_A^{\lambda}) + \beta_{L\lambda} (\overset{*\lambda}{w}_L - w_L^{\lambda})\} W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L \\
& + \sum_A \sum_B \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{f}_A + \overset{*}{f}_B - f_A - f_B - f_L) W_{ABL}\, dV_A\, dV_B\, d\overset{*}{V}_A\, d\overset{*}{V}_B \\
& + \sum_A \sum_B \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{\Delta}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B \{\beta_{A\lambda} (\overset{*\lambda}{w}_A - w_A^{\lambda}) + \beta_{B\lambda} (\overset{*\lambda}{w}_B - w_B^{\lambda}) - \beta_{L\lambda} w_L^{\lambda}\} W_{ABL}\, dV_A\, dV_B\, d\overset{*}{V}_A\, d\overset{*}{V}_B .
\end{aligned} \tag{7.17}$$

Since our discussion now becomes more complex, we shall discuss three cases. The first case is the consideration of $D_{\text{coll}}N_A$ ignoring the contribution $\tilde{D}_{\text{coll}}N_A$. Then we shall examine the contribution $\tilde{D}_{\text{coll}}N_A$. Lastly, we shall discuss the case of $D_{\text{coll}}N_L$.

B. Case One: Massive Particles Alone

We now consider equation (7.15) in which we now assume that $\tilde{D}_{coll}N_A$ is negligible. This means that we are only considering elastic binary collisions of massive particles with other massive particles. If we multiply (7.15) by a quantity $\xi_A=\xi_A(x,p^\alpha_A)$ and integrate we obtain

$$\begin{aligned}\int \xi_A D_{coll} N_A dV_A &= \sum_B \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{f}_A + \overset{*}{f}_B - f_A - f_B)\xi_A W_{AB} d^4V \\ &+ \sum_B \beta_{A\lambda} \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{w}{}^\lambda_A - w^\lambda_A)\xi_A W_{AB} d^4V \qquad (7.18) \\ &+ \sum_B \beta_{B\lambda} \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{w}{}^\lambda_B - w^\lambda_B)\xi_A W_{AB} d^4V \quad .\end{aligned}$$

If we rewrite the right hand side of (7.18) by exchanging starred and unstarred variables, employ equation (7.2), and then add the result to (7.18) we obtain

$$\begin{aligned}\int \xi_A D_{coll} N_A dV_A &= \frac{1}{2} \sum_B \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{f}_A + \overset{*}{f}_B - f_A - f_B)(\xi_A - \overset{*}{\xi}_A) W_{AB} d^4V \\ &+ \frac{1}{2} \sum_B \beta_{A\lambda} \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{w}{}^\lambda_A - w^\lambda_A)(\xi_A - \overset{*}{\xi}_A) W_{AB} d^4V \qquad (7.19) \\ &+ \frac{1}{2} \sum_B \beta_{B\lambda} \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{w}{}^\lambda_B - w^\lambda_B)(\xi_A - \overset{*}{\xi}_A) W_{AB} d^4V \quad .\end{aligned}$$

For massive particles, the structure of f_A, equation (4.2), implies that

$$\begin{aligned}\overset{*}{f}_A + \overset{*}{f}_B - f_A - f_B &= \tilde{b}_{A\lambda}(\overset{*}{w}{}^\lambda_A - w^\lambda_A) + \tilde{b}_{B\lambda}(\overset{*}{w}{}^\lambda_B - w^\lambda_B) \\ &+ \tilde{c}_{A\lambda\tau}(\overset{*}{w}{}^\lambda_A \overset{*}{w}{}^\tau_A - w^\lambda_A w^\tau_A) + \tilde{c}_{B\lambda\tau}(\overset{*}{w}{}^\lambda_B \overset{*}{w}{}^\tau_B - w^\lambda_B w^\tau_B) \quad .\end{aligned} \qquad (7.20)$$

Equations (7.19) and (7.20) now suggest that, for convenience, we should define an integral $C^{\alpha(n)}_{ABC}(\xi_A)$ by

$$C_{ABC\alpha(n)}(\xi_A) \equiv \frac{1}{2}\int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B \Big\{ \overset{*}{w}_{C\alpha_1} \cdots \overset{*}{w}_{C\alpha_n} - w_{C\alpha_1} \cdots w_{C\alpha_n} \Big\} (\overset{*}{\xi}_A - \xi_A) W_{AB} \, d^4V \quad . \tag{7.21}$$

This integral is symmetric in the indices $\alpha(n)$ and has the following properties:

$$g^{\lambda\tau} C_{ABC}(\xi_A)_{\alpha(n)\lambda\tau\beta(m)} = - C_{ABC}(\xi_A)_{\alpha(n)\beta(m)} \quad ;$$
$$g^{\lambda\tau} C_{ABC}(\xi_A)_{\lambda\tau} = 0 \; ; \; C_{ABC}(m_A \xi_A)_{\alpha(n)} = m_A C_{ABC}(\xi_A)_{\alpha(n)} \quad . \tag{7.22}$$

For the case $\alpha(n)=\alpha$ (one tensor index) we also have the following identity:

$$m_A C_{ABA\alpha}(\xi_A) + m_B C_{ABB\alpha}(\xi_A) = 0 \, , \tag{7.23}$$

which is based on the fact that the four-momentum is a summational invariant in collisions. When we substitute equation (7.20) into (7.18) and employ equation (7.21), we obtain an alternate expression for equation (7.18):

$$\int \xi_A D_{coll} N_A dV_A = -\sum_B \bar{b}^{\lambda}_A C_{ABA\lambda}(\xi_A) - \sum_B \bar{b}^{\lambda}_B C_{ABB\lambda}(\xi_A) - \sum_B \bar{c}^{\lambda\tau}_A C_{ABA\lambda\tau}(\xi_A) - \sum_B \bar{c}^{\lambda\tau}_B C_{ABB\lambda\tau}(\xi_A) - \sum_B \Big\{ \beta^{\lambda}_A C_{ABA\lambda}(\xi_A) - \beta^{\lambda}_B C_{ABB}(\xi_B)_{\lambda} \Big\} \quad . \tag{7.24}$$

A further simplification is obtained if we define

$$C_{AB\alpha(n)}(\xi_A) \equiv \begin{cases} C_{ABB\alpha(n)}(\xi_A) \ , \ A \neq B \\ \\ C_{AAA\alpha(n)}(\xi_A) + \sum\limits_{C} C_{ACA\alpha(n)}(\xi_A) \ , \ A = B \end{cases} . \qquad (7.25)$$

The identity (7.23) implies that

$$\sum_{B} m_B C_{AB\alpha}(\xi_A) = 0 \quad . \qquad (7.26)$$

Thus (7.24) becomes, via (7.25), the following expression:

$$\int \xi_A D_{coll} N_A dV_A = - \sum_{B} \Big\{ \bar{b}_B^{\lambda} C_{AB\lambda}(\xi_A) + \bar{c}_B^{\lambda\tau} C_{AB\lambda\tau}(\xi_A) + \beta_B^{\lambda} C_{AB\lambda}(\xi_A) \Big\} \quad . \qquad (7.27)$$

Two cases for ξ_A are of interest to us. These are $\xi_A = w_A^{\alpha}$ and $\xi_A = w_A^{\alpha} w_A^{\beta}$. To facilitate computation let us define the following four tensors:

$$L_{AB\lambda\tau} \equiv C_{AB\lambda}(w_{A\alpha}) \quad ; \qquad (7.28)$$

$$\chi_{AB\lambda\alpha\beta} \equiv C_{AB\lambda}(w_{A\alpha} w_{A\beta}) \quad ; \qquad (7.29)$$

$$L_{AB\lambda\tau\alpha} \equiv C_{AB\lambda\tau}(w_{A\alpha}) \quad ; \qquad (7.30)$$

$$\chi_{AB\lambda\tau\alpha\beta} \equiv C_{AB\lambda\tau}(w_{A\alpha} w_{A\beta}) \quad . \qquad (7.31)$$

These tensors have the following properties:

$$g^{\lambda\tau}L_{AB\alpha\lambda\tau} = 0 \; ; \; g^{\lambda\tau}\chi_{AB\lambda\tau\alpha} = 0 \; ; \; g^{\lambda\tau}\chi_{AB\lambda\tau\alpha\beta} = 0 \; ;$$
$$g^{\alpha\beta}\chi_{AB\lambda\tau\alpha\beta} = 0 \; ; \; L_{AB\lambda\alpha\beta} = L_{AB\lambda(\alpha\beta)} \; ; \qquad (7.32)$$
$$\chi_{AB\lambda\tau\alpha} = \chi_{AB(\lambda\tau)\alpha} \; ; \; \chi_{AB\lambda\tau\alpha\beta} = \chi_{AB(\lambda\tau)(\alpha\beta)} \; .$$

In terms of these new tensors, equation (7.27) gives for the two choices $\xi_A = w_A^\alpha$ and $\xi_A = w_A^\alpha w_A^\beta$ respectively that

$$T_{A\alpha}{}^{\lambda}{}_{|\lambda} = -\sum_B (\tilde{b}_B^{\lambda} L_{AB\lambda\alpha} + \tilde{c}_B^{\lambda\tau} L_{AB\lambda\tau\alpha} + \beta_B^{\lambda} L_{AB\lambda\alpha}) \quad ; \qquad (7.33)$$

$$U_{A\alpha\beta}{}^{\lambda}{}_{|\lambda} = -\sum_B (\tilde{b}_B^{\lambda}\chi_{AB\lambda\alpha\beta} + \tilde{c}_B^{\lambda\tau}\chi_{AB\lambda\tau\alpha\beta} + \beta_B^{\lambda}\chi_{AB\lambda\alpha\beta}) \; . \qquad (7.34)$$

To proceed further we must deduce the irreducible structures of the four tensors which are defined by equations (7.28) to (7.31). These tensors must be constructed out of scalars, the metric tensor, and the flow vector u^α. We note that our selection of a common flow vector for all species has greatly simplified the complexity of the tensor structures. We deduce the tensor structures by writing down the most general form that the tensors can have and then simplify by using the properties (7.32). Consequently, the irreducible tensor structures are given by

$$L_{AB\lambda\alpha} = L_{AB20}u_\lambda u_\alpha + L_{AB21}\Delta_{\lambda\alpha} \quad ; \qquad (7.35)$$

$$L_{AB\lambda\tau\alpha} = L_{AB30}(u_\lambda u_\tau + \tfrac{1}{3}\Delta_{\lambda\tau})u_\alpha + L_{AB31}(u_\lambda\Delta_{\tau\alpha} + u_\tau\Delta_{\lambda\alpha}) \;; \qquad (7.36)$$

$$\chi_{AB\lambda\alpha\beta} = \chi_{AB30}u_\lambda(u_\alpha u_\beta + \tfrac{1}{3}\Delta_{\alpha\beta}) + \chi_{AB31}(u_\alpha\Delta_{\lambda\beta} + u_\beta\Delta_{\lambda\alpha}) \;; \qquad (7.37)$$

$$\begin{aligned}\chi_{AB\lambda\tau\alpha\beta} = {} & \chi_{AB40}(u_\lambda u_\tau + \tfrac{1}{3}\,\Delta_{\lambda\tau})(u_\alpha u_\beta + \tfrac{1}{3}\,\Delta_{\alpha\beta}) \\ & + \chi_{AB41}(\Delta_{\lambda\alpha}u_\tau u_\beta + \Delta_{\lambda\beta}u_\tau u_\alpha + \Delta_{\tau\alpha}u_\lambda u_\beta + \Delta_{\tau\beta}u_\lambda u_\alpha) \qquad (7.38) \\ & + \chi_{AB42}(\Delta_{\lambda\alpha}\Delta_{\tau\beta} + \Delta_{\lambda\beta}\Delta_{\tau\alpha} - \tfrac{2}{3}\,\Delta_{\lambda\tau}\Delta_{\alpha\beta}) \;;\end{aligned}$$

where we have introduced an index notation to mimic the notation of I_{Anq} and J_{Anq} .

The coefficients L_{ABnq} and χ_{ABnq} which appear here can, in principle, be evaluated provided we know what the transition probabilities are, which in turn requires that we know what the collision cross sections are. Furthermore, the calculation of these coefficients depends upon the type of system being examined, for example, a binary mixture of electrons and protons versus, say, a mixture of electrons, protons, and neutrons. Let us therefore assume that these coefficients are, like the number and energy densities, known quantities.

In terms of the structures (7.35) to (7.38) we may rewrite equations (7.33) and (7.34) as

$$\begin{aligned}T_{A\alpha}{}^{\lambda}{}_{|\lambda} = -\sum_B \Big\{ & (L_{AB20}b_B + \tfrac{4}{3}L_{AB30}c_B - \beta_B L_{AB20})u_\alpha \\ & + L_{AB21}b_{B\alpha} + 2L_{AB31}c_{B\alpha} \Big\} \;; \qquad (7.39)\end{aligned}$$

$$U_{A\alpha\beta}{}^{\lambda}{}_{|\lambda} = -\sum_B \Bigg[(\chi_{AB30} b_B + \frac{4}{3}\chi_{AB40} c_B - \beta_B \chi_{AB30})(u_\alpha u_\beta + \frac{1}{3}\Delta_{\alpha\beta}) \tag{7.40}$$

$$+ 2\Big\{\chi_{AB31} b_{B(\alpha} + 2\chi_{AB41} c_{B(\alpha}\Big\} u_{\beta)} + 2\chi_{AB42} c_{B\alpha\beta} \Bigg] \ .$$

We now substitute into these two equation for b_B , c_B , etc., via equation (4.34), a process which leads to

$$T_A^{\alpha\lambda}{}_{|\lambda} = \Big\{ L_{A0} + \sum_B (L_{AB1}\delta n_B + L_{AB2}\delta\rho_B + L_{AB3}\pi_B) \Big\} u^\alpha \tag{7.41}$$

$$- \sum_B L_{AB4} h_B^\alpha - \sum_B L_{AB5} q_B^\alpha \quad ;$$

$$U_{A\alpha\beta}{}^{\lambda}{}_{|\lambda} = \Big\{ \tilde{\chi}_{A0} + \sum_B (\tilde{\chi}_{AB1}\delta n_B + \tilde{\chi}_{AB2}\delta\rho_B$$

$$+ \tilde{\chi}_{AB3}\pi_B) \Big\} (u_\alpha u_\beta + \frac{1}{3}\Delta_{\alpha\beta}) - \sum_B (2\tilde{\chi}_{AB4} h_{B(\alpha} u_{\beta)}) \tag{7.42}$$

$$- \sum_B (2\tilde{\chi}_{AB5} q_{B(\alpha} u_{\beta)}) - \sum_B \chi_{AB6} \pi_{B\alpha\beta} \quad ;$$

where we have defined

$$L_{A0} \equiv \sum_B \beta_B L_{AB30} \quad ; \tag{7.43}$$

$$L_{AB1} \equiv -(L_{AB20}\Omega_{B21} + \frac{4}{3} L_{AB30}\Omega_{B31}) \quad ; \tag{7.44}$$

$$L_{AB2} \equiv -(L_{AB20}\Omega_{B22} + \frac{4}{3} L_{AB30}\Omega_{B32}) \quad ; \tag{7.45}$$

$$L_{AB3} \equiv -(L_{AB20}\Omega_{B23} + \frac{4}{3} L_{AB30}\Omega_{B33}) \quad ; \tag{7.46}$$

$$L_{AB4} \equiv L_{AB21}\Omega_{B44} \quad ; \tag{7.47}$$

$$L_{AB5} \equiv L_{AB21}\Omega_{B45} + 2L_{AB31}\Omega_{B55} \quad ; \tag{7.48}$$

$$\tilde{\chi}_{A0} \equiv \sum_{B} \beta_B \chi_{AB30} \quad ; \tag{7.49}$$

$$\tilde{\chi}_{AB1} \equiv -\left(\chi_{AB30}\Omega_{B21} + \frac{4}{3}\,\chi_{AB40}\Omega_{B31}\right) \quad ; \tag{7.50}$$

$$\tilde{\chi}_{AB2} \equiv -\left(\chi_{AB30}\Omega_{B22} + \frac{4}{3}\,\chi_{AB40}\Omega_{B32}\right) \quad ; \tag{7.51}$$

$$\tilde{\chi}_{AB3} \equiv -\left(\chi_{AB30}\Omega_{B23} + \frac{4}{3}\,\chi_{AB40}\Omega_{B33}\right) \quad ; \tag{7.52}$$

$$\tilde{\chi}_{AB4} \equiv \chi_{AB31}\Omega_{B44} \quad ; \tag{7.53}$$

$$\tilde{\chi}_{AB5} \equiv \chi_{AB31}\Omega_{B45} + 2\chi_{AB41}\Omega_{B55} \quad ; \tag{7.54}$$

$$\chi_{AB6} \equiv 2\chi_{AB42}\Omega_{B66} \quad . \tag{7.55}$$

We note that if we choose the fitting conditions to give us a common inverse temperature $\tilde{\beta}$ then $L_{A0}=\tilde{\chi}_{A0}=0$ via the identity (7.26).

Equations (7.41) and (7.42) may be contracted with u^{α} and $\Delta^{\alpha\beta}$ in various ways to produce the following five convenient forms:

$$u_{\alpha}T^{\alpha\lambda}_{A}|\lambda = -L_{A0} - \sum_{B}(L_{AB1}\delta n_{B} + L_{AB2}\delta\rho_{B} + L_{AB3}\pi_{B}) \quad ; \qquad (7.56)$$

$$\Delta_{\mu\alpha}T^{\alpha\lambda}_{A}|\lambda = -\sum_{B}(L_{AB4}h_{B\mu} + L_{AB5}q_{B\mu}) \quad ; \qquad (7.57)$$

$$u_{\alpha}u_{\beta}U^{\alpha\beta\lambda}_{A}|\lambda = \tilde{\chi}_{A0} + \sum_{B}(\tilde{\chi}_{AB1}\delta n_{B} + \tilde{\chi}_{AB2}\delta\rho_{B} + \tilde{\chi}_{AB3}\pi_{B}) \quad ; \qquad (7.58)$$

$$\Delta_{\mu\alpha}u_{\beta}U^{\alpha\beta\lambda}_{A}|\lambda = \sum_{B}(\tilde{\chi}_{AB4}h_{B\mu} + \tilde{\chi}_{AB5}q_{B\mu}) \quad ; \qquad (7.59)$$

$$(\Delta_{\mu\alpha}\Delta_{\nu\beta} - \frac{1}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta})U^{\alpha\beta\lambda}_{A}|\lambda = -\sum_{B}\chi_{AB6}\pi_{B\mu\nu} \quad . \qquad (7.60)$$

These five equations are precisely those which are required to write the transport equations in their final form (chapter V).

C. Case Two: Massive and Massless Particle Interactions

The five equations immediately above do not include the interactions of the massive species A with massless particles. We now wish to know how they must be modified if we no longer consider $\tilde{D}_{coll}N_A$ to be negligible. To calculate this contribution we multiply $\tilde{D}_{coll}N_A$ by ξ_A and integrate. Denoting $d^5V \equiv d^4V dV_L$ we obtain

$$
\begin{aligned}
\int \xi_A \tilde{D}_{coll} N_A dV_A = {} & \sum_L \int \mathring{N}_A \mathring{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L (\overset{*}{f}_A + \overset{*}{f}_L - f_A - f_L) \xi_A W_{AL} d^4V \\
& + \sum_{LB}\sum \int \mathring{N}_A \mathring{N}_B \mathring{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{f}_A + \overset{*}{f}_B - f_A - f_B) \xi_A W_{ABL} d^5V \\
& + \sum_{LB}\sum \int \mathring{N}_A \mathring{N}_B \mathring{\Delta}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{f}_A + \overset{*}{f}_B - f_A - f_B) \xi_A W_{ABL} d^5V \\
& - \sum_{LB}\sum \int \mathring{N}_A \mathring{N}_B \mathring{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B f_L \xi_A W_{ABL} d^5V \\
& + \sum_{LB}\sum \int \mathring{N}_A \mathring{N}_B \mathring{\Delta}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B f_L \xi_A W_{ABL} d^5V \\
& + \sum_L \int \mathring{N}_A \mathring{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L \{\beta_{A\lambda} (\overset{*}{w}{}^{\lambda}_A - w^{\lambda}_A) + \beta_{L\lambda} (\overset{*}{w}{}^{\lambda}_L - w^{\lambda}_L)\} W_{AL} \xi_A d^4V \\
& + \sum_{LB}\sum \int \mathring{N}_A \mathring{N}_B \mathring{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B \{\beta_{A\lambda} (\overset{*}{w}{}^{\lambda}_A - w^{\lambda}_A) + \beta_{B\lambda} (\overset{*}{w}{}^{\lambda}_B - w^{\lambda}_B) - \beta_{L\lambda} w^{\lambda}_L\} \xi_A W_{ABL} d^5V \\
& + \sum_{LB}\sum \int \mathring{N}_A \mathring{N}_B \mathring{\Delta}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B \{\beta_{A\lambda} (\overset{*}{w}{}^{\lambda}_A - w^{\lambda}_A) + \beta_{B\lambda} (\overset{*}{w}{}^{\lambda}_B - w^{\lambda}_B) + \beta_{L\lambda} w^{\lambda}_L\} \xi_A W_{ABL} d^5V \; .
\end{aligned}
\tag{7.61}
$$

When we relabel variables in equation (7.61) by exchanging starred and unstarred variables, and employ equations (7.4) and (7.5), we obtain for (7.61) the following expression:

$$
\begin{aligned}
\int \xi_A \tilde{D}_{coll} N_A dV_A = {} & \frac{1}{2} \sum_L \int \mathring{N}_A \mathring{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L (\overset{*}{f}_A - f_A)(\xi_A - \overset{*}{\xi}_A) W_{AL} d^4V \\
& + \sum_L \int \mathring{N}_A \mathring{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L f_L (\xi_A - \overset{*}{\xi}_A) W_{AL} d^4V \\
& + \frac{1}{2} \sum_{LB}\sum \int \mathring{N}_A \mathring{N}_B \mathring{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{f}_A + \overset{*}{f}_B - f_A - f_B)(\xi_A - \overset{*}{\xi}_A) W_{ABL} d^5V \\
& + \text{(next page)}
\end{aligned}
$$

$$\begin{aligned}
&+\frac{1}{2}\sum_{LB}\sum\int \overset{\circ}{N}_A\overset{\circ}{N}_B\overset{\circ}{\Delta}_L\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_B(\overset{*}{f}_A+\overset{*}{f}_B-f_A-f_B)(\xi_A-\overset{*}{\xi}_A)W_{ABL}\,d^5V \\
&-\frac{1}{2}\sum_{LB}\sum\int \overset{\circ}{N}_A\overset{\circ}{N}_B\overset{\circ}{N}_L\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_B f_L(\xi_A-\overset{*}{\xi}_A)W_{ABL}\,d^5V \\
&+\frac{1}{2}\sum_{LB}\sum\int \overset{\circ}{N}_A\overset{\circ}{N}_B\overset{\circ}{\Delta}_L\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_B f_L(\xi_A-\overset{*}{\xi}_A)W_{ABL}\,d^5V \\
&+\frac{1}{2}\sum_{LB}\sum\int \overset{\circ}{N}_A\overset{\circ}{N}_L\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_L\{\beta_{A\lambda}(\overset{*}{w}{}^{\lambda}_A-w^{\lambda}_A)+\beta_{L\lambda}(\overset{*}{w}{}^{\lambda}_L-w^{\lambda}_L)\}(\xi_A-\overset{*}{\xi}_A)W_{AL}\,d^4V \\
&+\frac{1}{2}\sum_{LB}\sum\int \overset{\circ}{N}_A\overset{\circ}{N}_B\overset{\circ}{N}_L\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_B\{\beta_{A\lambda}(\overset{*}{w}{}^{\lambda}_A-w^{\lambda}_A)+\beta_{B\lambda}(\overset{*}{w}{}^{\lambda}_B-w^{\lambda}_B)-\beta_{L\lambda}w^{\lambda}_L\}(\xi_A-\overset{*}{\xi}_A)W_{ABL}\,d^5V \\
&+\frac{1}{2}\sum_{LB}\sum\int \overset{\circ}{N}_A\overset{\circ}{N}_B\overset{\circ}{\Delta}_L\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_B\{\beta_{A\lambda}(\overset{*}{w}{}^{\lambda}_A-w^{\lambda}_A)+\beta_{B\lambda}(\overset{*}{w}{}^{\lambda}_B-w^{\lambda}_B)+\beta_{L\lambda}w^{\lambda}_L\}(\xi_A-\overset{*}{\xi}_A)W_{ABL}\,d^5V\ .
\end{aligned} \tag{7.62}$$

In light of our previous arguments in this chapter, we define four integrals which are given by

$$\begin{aligned}
C_{ALC\alpha(n)}(\xi_A) \equiv \frac{1}{2}\int \overset{\circ}{N}_A\overset{\circ}{N}_L\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_L(&\overset{*}{w}_{C\alpha_1}\cdots\overset{*}{w}_{C\alpha_n} \\
&- w_{C\alpha_1}\cdots w_{C\alpha_n})(\overset{*}{\xi}_A-\xi_A)W_{AL}\,d^4V\quad ;
\end{aligned} \tag{7.63}$$

$$\begin{aligned}
C^1_{LABC\alpha(n)}(\xi_A) \equiv \frac{1}{2}\int \overset{\circ}{N}_A\overset{\circ}{N}_B(\overset{\circ}{N}_L+\overset{\circ}{\Delta}_L)\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_B(&\overset{*}{w}_{C\alpha_1}\cdots\overset{*}{w}_{C\alpha_n} \\
&- w_{C\alpha_1}\cdots w_{C\alpha_n})(\overset{*}{\xi}_A-\xi_A)W_{ABL}\,d^5V\quad ;
\end{aligned} \tag{7.64}$$

$$C^{2\ \alpha(n)}_{LAB}(\xi_A) \equiv \frac{1}{2}\int \overset{\circ}{N}_A\overset{\circ}{N}_B\overset{\circ}{N}_L\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_B(w^{\alpha_1}_L\ldots w^{\alpha_n}_L)(\overset{*}{\xi}_A-\xi_A)W_{ABL}\,d^5V\quad ; \tag{7.65}$$

$$C^{3\ \alpha(n)}_{LAB}(\xi_A) \equiv \frac{1}{2}\int \overset{\circ}{N}_A\overset{\circ}{N}_B\overset{\circ}{\Delta}_L\overset{\circ *}{\Delta}_A\overset{\circ *}{\Delta}_B(w^{\alpha_1}_L\ldots w^{\alpha_n}_L)(\overset{*}{\xi}_A-\xi_A)W_{ABL}\,d^5V\quad . \tag{7.66}$$

When we substitute equation (7.20) into (7.62) and employ the integrals just defined, equation (7.62) becomes

$$\begin{aligned}\int \xi_A \tilde{D}_{coll} N_A dV_A = &- \sum_L \left\{ \tilde{b}_A^{\lambda} C_{ALA\lambda}(\xi_A) + \tilde{c}_A^{\lambda\tau} C_{ALA\lambda\tau}(\xi_A) \right\} \\ &- \sum_{LB}\sum \left\{ \tilde{b}_A^{\lambda} C^1_{LABA\lambda}(\xi_A) + \tilde{c}_A^{\lambda\tau} C^1_{LABA\lambda\tau}(\xi_A) \right. \\ &\qquad \left. + \tilde{b}_B^{\lambda} C^1_{LABB\lambda}(\xi_A) + \tilde{c}_B^{\lambda\tau} C^1_{LABB\lambda\tau}(\xi_A) \right\} \\ &- \sum_L \left\{ \beta_A^{\lambda} C_{ALA\lambda}(\xi_A) + \beta_L^{\lambda} C_{ALL\lambda}(\xi_A) \right\} \\ &- \sum_{LB}\sum \left\{ \beta_A^{\lambda} C^1_{LABA\lambda}(\xi_A) + \beta_B^{\lambda} C^1_{LABB\lambda} - \beta_L^{\lambda} C^2_{LAB\lambda}(\xi_A) \right. \\ &\left. + \beta_L^{\lambda} C^3_{LAB\lambda}(\xi_A) \right\} - \sum_L \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L (\overset{*}{\xi}_A - \xi_A) f_L W_{AL} d^4V \\ &+ \frac{1}{2} \sum_{LB}\sum \int \overset{\circ}{N}_A \overset{\circ}{N}_B (\overset{\circ}{\Delta}_L - \overset{\circ}{N}_L) \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{\xi}_A - \xi_A) f_L W_{ABL} d^5V \quad . \end{aligned} \tag{7.67}$$

We may simplify equation (7.67) by defining

$$C^{1\ \alpha(n)}_{AC}(\xi_A) \equiv \begin{cases} \sum\limits_L C^{\alpha(n)}_{ALA}(\xi_A) + \sum\limits_{LB}\sum C^{1\ \alpha(n)}_{LABA}(\xi_A) & ;\ C = A \\ \sum\limits_L C^{1\ \alpha(n)}_{LABB}(\xi_A) & ;\ C = B\ ,\ C \neq A,L \\ C^{\alpha(n)}_{ALL}(\xi_A) - \sum\limits_B C^{2\ \alpha(n)}_{LAB}(\xi_A) + \sum\limits_B C^{3\ \alpha(n)}_{LAB}(\xi_A) & ;C=L \end{cases} \tag{7.68}$$

in terms of which equation (7.67) takes on the following form:

$$\begin{aligned}\int \xi_A \tilde{D}_{coll} N_A dV_A = &- \sum_B \left\{ \tilde{b}_B^{\lambda} C^1_{AB\lambda}(\xi_A) + \tilde{c}_B^{\lambda\tau} C_{AB\lambda\tau}(\xi_A) \right\} \\ &- \sum_C \beta_C^{\lambda} C^1_{AC\lambda}(\xi_A) \\ &+ \sum_L \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L f_L (\xi_A - \overset{*}{\xi}_A) W_{AL} d^4V \\ &+ \frac{1}{2} \sum_{LB}\sum \int \overset{\circ}{N}_A \overset{\circ}{N}_B (\overset{\circ}{\Delta}_L - \overset{\circ}{N}_L) \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B f_L (\xi_A - \overset{*}{\xi}_A) W_{ABL} d^5V \quad . \end{aligned} \tag{7.69}$$

As before, we are primarily interested in the two cases $\xi_A = w_A^\alpha$ and $\xi_A = w_A^\alpha w_A^\beta$. We notice, however, that the first four terms in equation (7.69) are similar to equation (7.27). Hence, we just have to repeat the subsequent analysis from (7.27) onward, merely adding a superscript 1 to the quantities defined therein. Then, if we denote the contributions to $T_{A|\lambda}^{\alpha\lambda}$ and $U_{A\ |\lambda}^{\alpha\beta\lambda}$ by $T_{A\ |\lambda}^{1\alpha\lambda}$ and $U_{A\ \ |\lambda}^{1\alpha\beta\lambda}$ we obtain

$$\begin{aligned} T^{1\ \lambda}_{A\alpha\ |\lambda} = {} & \Big\{L^1_{A0} + \sum_B (L^1_{AB1}\delta n_B + L^1_{AB2}\delta\rho_B + L^1_{AB3}\pi_B)\Big\}u_\alpha \\ & - \sum_B L^1_{AB4}h_{B\alpha} - \sum_B L^1_{AB5}q_{B\alpha} \\ & + \sum_L \int \mathring{N}_A \mathring{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L f_L (w_{A\alpha} - \overset{*}{w}_{A\alpha}) W_{AL}\, d^4V \\ & + \frac{1}{2} \sum_{LB}\sum \int \mathring{N}_A \mathring{N}_B (\mathring{\Delta}_L - \mathring{N}_L) \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B f_L (w_{A\alpha} - \overset{*}{w}_{A\alpha}) W_{ABL}\, d^5V \quad ; \end{aligned} \tag{7.70}$$

$$\begin{aligned} U^{1\ \ \lambda}_{A\alpha\beta|\lambda} = {} & \Big\{\tilde{\chi}^1_{A0} + \sum_B (\tilde{\chi}^1_{AB1}\delta n_B + \tilde{\chi}^1_{AB2}\delta\rho_B + \tilde{\chi}^1_{AB3}\pi_B\Big\}(u_\alpha u_\beta + \tfrac{1}{3}\Delta_{\alpha\beta}) \\ & - \sum_B 2\tilde{\chi}^1_{AB4}h_{B(\alpha}u_{\beta)} - \sum_B 2\tilde{\chi}^1_{AB5}q_{B(\alpha}u_{\beta)} - \sum_B \chi^1_{AB6}\pi_{B\alpha\beta} \\ & + \sum_L \int \mathring{N}_A \mathring{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L f_L (w_{A\alpha}w_{A\beta} - \overset{*}{w}_{A\alpha}\overset{*}{w}_{A\beta}) W_{AL}\, d^4V \\ & + \frac{1}{2} \sum_{LB}\sum \int \mathring{N}_A \mathring{N}_B (\mathring{\Delta}_L - \mathring{N}_L) \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B f_L (w_{A\alpha}w_{A\beta} - \overset{*}{w}_{A\alpha}\overset{*}{w}_{A\beta}) W_{ABL}\, d^5V \quad . \end{aligned} \tag{7.71}$$

We must now consider the structures of the integrals containing f_L in equations (7.70) and (7.71). When we integrate over all variables except the variable p_L^α in the four integrals in (7.70) and (7.71), the results must be a tensor function of position and p_L^α. Consequently we have that

$$\int \mathring{N}_A \mathring{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L (w_{A\alpha} - \overset{*}{w}_{A\alpha}) W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L$$

$$+ \frac{1}{2} \sum_B \int \mathring{N}_A \mathring{N}_B (\mathring{\Delta}_L - \mathring{N}_L) \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (w_{A\alpha} - \overset{*}{w}_{A\alpha}) W_{ABL}\, dV_A\, d\overset{*}{V}_A\, dV_B\, d\overset{*}{V}_B \tag{7.72}$$

$$= \frac{1}{4\pi} J_{L10}(\nu_L) \left\{ T^1_{AL} u_\alpha - T^2_{AL} k_{L\alpha} \right\} \;;$$

$$\int \mathring{N}_A \mathring{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L (w_{A\alpha} w_{A\beta} - \overset{*}{w}_{A\alpha} \overset{*}{w}_{A\beta}) W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L$$

$$+ \frac{1}{2} \sum_B \int \mathring{N}_A \mathring{N}_B (\mathring{\Delta}_L - \mathring{N}_L) \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (w_{A\alpha} w_{A\beta} - \overset{*}{w}_{A\alpha} \overset{*}{w}_{A\beta}) W_{ABL}\, dV_A\, d\overset{*}{V}_A\, dV_B\, d\overset{*}{V}_B \tag{7.73}$$

$$= \frac{1}{4\pi} J_{L10}(\nu_L) \left\{ U^1_{AL} (u_\alpha u_\beta + k_{L\alpha} k_{L\beta}) - U^2_{AL} (u_\alpha k_{L\beta} + u_\beta k_{L\alpha}) \right\}$$

where k_L^α is given by equation (6.5). The coefficients which appear in (7.72) and (7.73) depend, since they are scalars, upon position and the frequency ν_L.

We multiply (7.72) and (7.73) by f_L as given by equation (6.25) and integrate over dV_L, thereby obtaining for the four integrals in question the following expressions:

$$\int \mathring{N}_A \mathring{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L f_L (w_{A\alpha} - \overset{*}{w}_{A\alpha}) W_{AL}\, d^4V$$

$$+ \frac{1}{2} \sum_B \int \mathring{N}_A \mathring{N}_B (\mathring{\Delta}_L - \mathring{N}_L) \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B f_L (w_{A\alpha} - \overset{*}{w}_{A\alpha}) W_{ABL}\, d^5V \tag{7.74}$$

$$= L^1_{AL1} \delta\rho_L u_\alpha - L^1_{AL4} h_{L\alpha} \;;$$

$$\int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L f_L (w_{A\alpha} w_{A\beta} - \overset{*}{w}_{A\alpha} \overset{*}{w}_{A\beta}) W_{AL} d^4V$$

$$+ \frac{1}{2} \sum_B \int \overset{\circ}{N}_A \overset{\circ}{N}_B (\overset{\circ}{\Delta}_L - \overset{\circ}{N}_L) \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B f_L (w_{A\alpha} w_{A\beta} - \overset{*}{w}_{A\alpha} \overset{*}{w}_{A\beta}) W_{ABL} d^5V \qquad (7.75)$$

$$= \tilde{\chi}^1_{AL2} \delta\rho_L (u_\alpha u_\beta + \frac{1}{3} \Delta_{\alpha\beta}) - \tilde{\chi}^1_{AL4} (h_{L\alpha} u_\beta + h_{L\beta} u_\alpha) - \chi^1_{AL6} \pi_{L\alpha\beta} \quad ;$$

where the average coefficients L^1_{ALi} and $\tilde{\chi}^1_{ALi}$ are defined by

$$L^1_{AL2} \delta\rho_L \equiv \int T^1_{AL} \delta\rho_L (\nu_L) d\nu_L \quad ; \quad L^1_{AL4} h_{L\alpha} \equiv \int T^2_{AL} h_{L\alpha} (\nu_L) d\nu_L ; \qquad (7.76)$$

$$\tilde{\chi}^1_{AL2} \delta\rho_L \equiv \int U^1_{AL} \delta\rho_L (\nu_L) \, d\nu_L \quad ;$$

$$\tilde{\chi}^1_{AL4} h_{L\alpha} \equiv \int U^2_{AL} h_{L\alpha} (\nu_L) \, d\nu_L \quad ; \qquad (7.77)$$

$$\chi_{AL6} \pi_{L\alpha\beta} \equiv - \int U^1_{AL} \pi_{L\alpha\beta} (\nu_L) \, d\nu_L \quad .$$

Strictly speaking, L^1_{AL4} , $\tilde{\chi}^1_{AL4}$, and $\chi^1{}_{AL6}$ should be tensors but we have assumed they are scalars for simplicity [2].

Substituting (7.76) and (7.77) into equations (7.70) and (7.71) now gives us

$$T^{1\ \lambda}_{A\alpha\ |\lambda} = \left\{ L^1_{A0} + \sum_B (L^1_{AB1} \delta n_B + L^1_{AB2} \delta\rho_B + L^1_{AB3} \pi_B) \right\} u_\alpha$$

$$- \sum_B L^1_{AB4} h_{B\alpha} - \sum_B L^1_{AB5} q_{B\alpha} \qquad (7.78)$$

$$+ \sum_L L^1_{AL2} \delta\rho_L - \sum_L L^1_{AL4} h_{L\alpha} \quad ;$$

$$
\begin{aligned}
U^{1\ \ \ \lambda}_{A\alpha\beta\ |\lambda} = \Big\{ & \tilde{\chi}^1_{A0} + \sum_B \big(\tilde{\chi}^1_{AB1}\delta n_B + \tilde{\chi}^1_{AB2}\delta\rho_B \\
& + \tilde{\chi}^1_{AB3}\pi_B \big) + \sum_L \tilde{\chi}^1_{AL2}\delta\rho_L \Big\} \big(u_\alpha u_\beta + \tfrac{1}{3}\Delta_{\alpha\beta}\big) \\
& - 2\Big\{ \sum_B \tilde{\chi}^1_{AB4} h_{B(\alpha}u_{\beta)} + \sum_L \tilde{\chi}^1_{AL4} h_{L(\alpha}u_{\beta)} \Big\} \\
& - 2\sum_B \tilde{\chi}^1_{AB5} q_{B(\alpha}u_{\beta)} - \sum_B \chi^1_{AB6}\pi_{B\alpha\beta} - \sum_L \chi^1_{AL6}\pi_{L\alpha\beta} \quad .
\end{aligned}
\tag{7.79}
$$

Various contractions with u^α and $\Delta^{\alpha\beta}$ produce the five equations which are the contributions to $T^{\alpha\lambda}_{A|\lambda}$ and $U^{\alpha\beta\lambda}_{A\ |\lambda}$ due to interactions with massless particles:

$$
-u_\alpha T^{1\alpha\lambda}_{A\ |\lambda} = L^1_{A0} + \sum_B \big(L^1_{AB1}\delta n_B + L^1_{AB2}\delta\rho_B + L^1_{AB3}\pi_B\big) + \sum_L L^1_{AL2}\delta\rho_L \quad ; \tag{7.80}
$$

$$
-\Delta_{\mu\alpha} T^{1\alpha\lambda}_{A\ |\lambda} = \sum_B L^1_{AB4} h_{B\mu} + \sum_L L^1_{AL4} h_{L\mu} + \sum_B L^1_{AB5} q_{B\mu} \quad ; \tag{7.81}
$$

$$
u_\alpha u_\beta U^{1\alpha\beta\lambda}_{A\ \ |\lambda} = \tilde{\chi}^1_{A0} + \sum_B \big(\tilde{\chi}^1_{AB1}\delta n_B + \tilde{\chi}^1_{AB2}\delta\rho_B + \tilde{\chi}^1_{AB3}\pi_B\big) + \sum_L \tilde{\chi}^1_{AL2}\delta\rho_L \quad ; \tag{7.82}
$$

$$
\Delta_{\mu\alpha} u_\beta U^{1\alpha\beta\lambda}_{A\ \ |\lambda} = \sum_B \tilde{\chi}^1_{AB4} h_{B\mu} + \sum_L \tilde{\chi}^1_{AL4} h_{L\mu} + \sum_B \tilde{\chi}^1_{AB5} q_{B\mu} \quad ; \tag{7.83}
$$

$$
\big(\Delta_{\mu\alpha}\Delta_{\nu\beta} - \tfrac{1}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta}\big) U^{1\alpha\beta\lambda}_{A\ \ |\lambda} = -\sum_B \chi^1_{AB6}\pi_{B\mu\nu} - \sum_L \chi^1_{AL6}\pi_{L\mu\nu} \ . \tag{7.84}
$$

D. Case Three: Massless Particles Alone

The last case we have to consider is the evaluation of $D_{coll}N_L$. Let us reproduce equation (7.17) in a more convenient arrangement:

$$\begin{aligned}
D_{coll}N_L = {} & \sum_A \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L (\overset{*}{f}_A - f_A) W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L \\
& + \sum_{AB}\sum \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{f}_A - f_A) W_{ABL}\, d^4V \\
& + \sum_{AB}\sum \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{f}_B - f_B) W_{ABL}\, d^4V \\
& + \sum_A \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L \overset{*}{f}_L W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L \qquad (7.85) \\
& - \overset{\circ}{N}_L f_L \Big\{ \sum_A \int \overset{\circ}{N}_A \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L \\
& \qquad + \sum_{AB}\sum \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B W_{ABL}\, d^4V \Big\} \\
& + \sum_A \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L \{ \beta_{A\lambda} (\overset{*}{w}{}^{\lambda}_A - w^{\lambda}_A) + \beta_{L\lambda} (\overset{*}{w}{}^{\lambda}_L - w^{\lambda}_L) \} W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L \\
& + \sum_{AB}\sum \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{\Delta}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B \{ \beta_{A\lambda} (\overset{*}{w}{}^{\lambda}_A - w^{\lambda}_A) + \beta_{B\lambda} (\overset{*}{w}{}^{\lambda}_B - w^{\lambda}_B) - \beta_{L\lambda} w^{\lambda}_L \} W_{ABL}\, d^4V \quad .
\end{aligned}$$

The structure of f_A for massive particles suggests that we define the following integrals:

$$K^{\alpha(n)}_{LA}(p^{\alpha}_L) \equiv 4\pi\nu^2_L \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L (\overset{*}{w}{}^{\alpha_1}_A \ldots w^{\alpha_n}_A - w^{\alpha_1}_A \ldots w^{\alpha_n}_A) W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L \quad ; \qquad (7.86)$$

$$K^{\alpha(n)}_{LABC}(p^{\alpha}_L) \equiv 4\pi\nu^2_L \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B (\overset{*}{w}{}^{\alpha_1}_C \ldots \overset{*}{w}{}^{\alpha_n}_C - w^{\alpha_1}_C \ldots w^{\alpha_n}_C) W_{ABL}\, d^4V \quad ; \qquad (7.87)$$

$$K_L(p_L^{\alpha}) \equiv 4\pi\nu_L^2\left\{\sum_A\int \overset{\circ}{N}_A \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L \right. \qquad (7.88)$$

$$\left. + \sum_{AB}\sum\int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B W_{ABL}\, d^4V\right\} \quad ;$$

$$\tilde{K}_{LL\lambda} \equiv - 4\pi\nu_L^2 \sum_{AB}\sum \int \overset{\circ}{N}_A \overset{\circ}{N}_B \overset{\circ}{\Delta}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_B w_{L\lambda} W_{ABL}\, d^4V$$

$$+ 4\pi\nu_L^2 \sum_A \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L (\overset{*}{w}_{L\lambda} - w_{L\lambda}) W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L \quad . \qquad (7.89)$$

Then equation (7.85) becomes in terms of these integrals and the structure of f_A the following expression:

$$4\pi\nu_L^2 D_{coll} N_L = \sum_A (\tilde{b}_A^{\lambda} K_{LA\lambda} + \tilde{c}_A^{\lambda\tau} K_{LA\lambda\tau})$$

$$+ \sum_{AB}\sum \left\{ \tilde{b}_A^{\lambda} K_{LABA\lambda} + \tilde{c}_A^{\lambda\tau} K_{LABA\lambda\tau} \right.$$

$$\left. + \tilde{b}_B^{\lambda} K_{LABB\lambda} + \tilde{c}_B^{\lambda\tau} K_{LABB\lambda\tau} \right\} \qquad (7.90)$$

$$+ \sum_{AB}\sum \left\{ \beta_A^{\lambda} K_{LA\lambda} + \beta_A^{\lambda} K_{LABA\lambda} + \beta_B^{\lambda} K_{LABB\lambda} \right\}$$

$$- 4\pi\nu_L^2 K_L \overset{\circ}{N}_L f_L + \beta_L^{\lambda} \tilde{K}_{LL\lambda}$$

$$+ 4\pi\nu_L^2 \sum_A \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L \overset{*}{f}_L W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L$$

If we define

$$\tilde{K}_{LC}^{\alpha(n)} \equiv \begin{cases} \tilde{K}_{LL}^{\alpha} \text{ , } \alpha(n) = \alpha \text{ , } C = L \quad ; \\ K_{LA}^{\alpha(n)} + \sum_B K_{LABA}^{\alpha(n)} + \sum_B K_{LBAA}^{\alpha(n)} \text{ , } C = A \quad ; \end{cases} \qquad (7.91)$$

then equation (7.90) simplifies to

$$
\begin{aligned}
4\pi\nu_L^2 D_{\mathrm{coll}} N_L = \sum_A \left\{ \tilde{b}_A^{\lambda} \tilde{K}_{LA\lambda} + \tilde{c}_A^{\lambda\tau} \tilde{K}_{LA\lambda\tau} \right\} \\
+ \sum_A \beta_A^{\lambda} \tilde{K}_{LA\lambda} + \beta_L^{\lambda} \tilde{K}_{LL\lambda} - 4\pi\nu_L^2 \overset{\circ}{N}_L f_L K_L \qquad (7.92) \\
+ 4\pi\nu_L^2 \sum_A \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ*}{\Delta}_A \overset{\circ*}{\Delta}_L \overset{*}{f}_L W_{AL} dV_A d\overset{*}{V}_A d\overset{*}{V}_L \quad .
\end{aligned}
$$

To make further progress we must express $\tilde{K}_{LC\lambda}$ and $\tilde{K}_{LC\lambda\tau}$ in terms of their irreducible structures. Since they are tensor functions of position, momentum and the flow vector u^α , we obtain

$$
\begin{aligned}
\tilde{K}_{LC\alpha} = \tilde{K}_{LC}^1 u_\alpha + \tilde{K}_{LC}^2 k_{L\alpha} \quad ; \\
\tilde{K}_{LA\lambda\tau} = \tilde{K}_{LA}^3 (u_\lambda u_\tau + k_{L\lambda} k_{L\tau}) + \tilde{K}_{LA}^4 (u_\lambda u_\tau + \tfrac{1}{3} \Delta_{\lambda\tau}) \qquad (7.93) \\
+ \tilde{K}_{LA}^5 (u_\lambda k_{L\tau} + u_\tau k_{L\lambda}) \quad .
\end{aligned}
$$

Substituting these expressions into (7.92) we then have that

$$
\begin{aligned}
4\pi\nu_L^2 D_{\mathrm{coll}} N_L = \sum_A \left\{ \tilde{K}_{LA}^1 b_A + \tilde{K}_{LA}^2 b_A^\alpha k_{L\alpha} + \frac{4}{3} c_A (\tilde{K}_{LA}^3 + \tilde{K}_{LA}^4) \right. \\
\left. + 2\tilde{K}_{LA}^5 c_A^\alpha k_{L\alpha} + \tilde{K}_{LA}^3 c_A^{\alpha\beta} k_{L\alpha} k_{L\beta} \right\} \qquad (7.94) \\
- \sum_A \beta_A \tilde{K}_{LA}^1 - \beta_L \tilde{K}_{LL}^1 - 4\pi\nu_L^2 K_L \overset{\circ}{N}_L f_L \\
+ 4\pi\nu_L^2 \sum_A \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ*}{\Delta}_A \overset{\circ*}{\Delta}_L \overset{*}{f}_L W_{AL} dV_A d\overset{*}{V}_A d\overset{*}{V}_L \quad .
\end{aligned}
$$

Let us re-express b_A , c_A, etc. via equation (4.34). If we define

$$
M_{LA1} \equiv - \left\{ \tilde{K}_{LA}^1 \Omega_{A21} + \frac{4}{3} (\tilde{K}_{LA}^3 + \tilde{K}_{LA}^4) \Omega_{A31} \right\} \quad ; \qquad (7.95)
$$

$$M_{LA2} \equiv -\left\{ \tilde{K}^1_{LA}\Omega_{A22} + \frac{4}{3}\,(\tilde{K}^3_{LA} + \tilde{K}^4_{LA})\Omega_{A32} \right\} \quad ; \tag{7.96}$$

$$M_{LA3} \equiv -\left\{ \tilde{K}^1_{LA}\Omega_{A23} + \frac{4}{3}\,(\tilde{K}^3_{LA} + \tilde{K}^4_{LA})\Omega_{A33} \right\} \quad ; \tag{7.97}$$

$$M_{LC0} \equiv \tilde{K}^1_{LC} \quad , \quad C = A,\, L \quad ; \tag{7.98}$$

$$M_{LA4} \equiv -\tilde{K}^2_{LA}\Omega_{A44} \div 3 \quad ; \tag{7.99}$$

$$M_{LA5} \equiv -\left\{ \tilde{K}^2_{LA}\Omega_{A45} + 2\tilde{K}^5_{LA}\Omega_{A55} \right\} \div 3 \quad ; \tag{7.100}$$

$$M_{LA6} \equiv -\frac{2}{15}\,\tilde{K}^3_{LA}\Omega_{A66} \quad ; \tag{7.101}$$

then we have for equation (7.94) the following result:

$$\begin{aligned} 4\pi\nu^2_L D_{coll} N_L = -\sum_A \Big\{ & M_{LA1}\delta n_A + M_{LA2}\delta\rho_A + M_{LA3}\pi_A \\ & +3\, M_{LA4} h^{\alpha}_A k_{L\alpha} + 3\, M_{LA5} q^{\alpha}_A k_{L\alpha} \\ & + \frac{15}{2} M_{LA6}\pi^{\alpha\beta}_A k_{L\alpha} k_{L\beta} \Big\} - \sum_C M_{LC0}\beta_C \\ & - 4\pi\nu^2_L K_L \overset{\circ}{N}_L f_L + 4\pi\nu^2_L \sum_A \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ}{\Delta}{}^*_A \overset{\circ}{\Delta}{}^*_L \overset{*}{f}_L W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L \quad . \end{aligned} \tag{7.102}$$

A simplification occurs if we define

$$M_{L0} \equiv \sum_{C} M_{LC0}\beta_C \quad . \tag{7.103}$$

This quantity is zero if all of the inverse temperatures, $\tilde{\beta}_C$, ar the same.

We must now determine how to evaluate the last integral in equation (7.102). Since we are integrating over the variables p^{α}_{A} , $\overset{*}{p}{}^{\alpha}_{A}$, $\overset{*}{p}{}^{\alpha}_{L}$, and in light of the structure of f_L , equation (6.25), this integral must be a function solely of the variables p^{α}_{L} and position. We conclude that [35]

$$\begin{aligned} &4\pi\nu_L^2 \int \overset{\circ}{N}_A \overset{\circ}{N}_L \overset{\circ *}{\Delta}_A \overset{\circ *}{\Delta}_L \overset{*}{f}_L W_{AL}\, dV_A\, d\overset{*}{V}_A\, d\overset{*}{V}_L \\ &= - K^{-1}_{LA}\delta\rho_L(\nu_L) - K^{-2}_{LA} h^{\alpha}_{L}(\nu_L) k_{L\alpha} - K^{-3}_{LA}\pi^{\alpha\beta}_{L}(\nu_L) k_{L\alpha} k_{L\beta} \ ; \end{aligned} \tag{7.104}$$

where the coefficients K^{-1}_{LA} must depend solely upon position and frequency.

Equation (7.102) becomes in terms of (7.103) and (7.104) the following expression:

$$\begin{aligned} - 4\pi\nu_L^2 D_{\mathrm{coll}} N_L = M_{L0} &+ \sum_{A} \Big\{ M_{LA1}\delta n_A + M_{LA2}\delta\rho_A + M_{LA3}\pi_A \\ &+ 3M_{LA4} h^{\alpha}_{A} k_{L\alpha} + 3M_{LA5} q^{\alpha}_{A} k_{L\alpha} \\ &+ \frac{15}{2} M_{LA6}\pi^{\alpha\beta}_{A} k_{L\alpha} k_{L\beta} \Big\} + 4\pi\nu_L^2 K_L \overset{\circ}{N}_L f_L \\ &+ \sum_{A} \Big\{ K^{-1}_{LA}\delta\rho_L + K^{-2}_{LA} h^{\alpha}_{L} k_{L\alpha} + K^{-3}_{LA}\pi^{\alpha\beta}_{L} k_{L\alpha} k_{L\beta} \Big\} \quad . \end{aligned} \tag{7.105}$$

We express the remaining f_L in this equation by equation

(6.24). Hence, when we define

$$\kappa_{L1} \equiv \{4\pi\nu_L^2 K_L \overset{\circ}{N}_L / J_{L20}(\nu_L)\} + \sum_A K_{LA}^{-1} \quad ; \tag{7.106}$$

$$\kappa_{L2} \equiv \{4\pi\nu_L^2 K_L \overset{\circ}{N}_L / J_{L20}(\nu_L)\} + \frac{1}{3} \sum_A K_{LA}^{-2} \quad ; \tag{7.107}$$

$$\kappa_{L3} \equiv \{4\pi\nu_L^2 K_L \overset{\circ}{N}_L / J_{L20}(\nu_L)\} + \frac{2}{15} \sum_A K_{LA}^{-3} \quad ; \tag{7.108}$$

we obtain

$$4\pi\nu_L^2 D_{coll} N_L = \tilde{D}_L + \tilde{D}_L^{\alpha} k_{L\alpha} + \tilde{D}_L^{\alpha\beta} k_{L\alpha} k_{L\beta} \quad ; \tag{7.109}$$

where we have defined the following coefficients:

$$\tilde{D}_L \equiv - \left\{ M_{L0} + \kappa_{L1} \delta\rho_L + \sum_A (M_{LA1} \delta n_A + M_{LA2} \delta\rho_A + M_{LA3} \pi_A) \right\} ; \tag{7.110}$$

$$\tilde{D}_L^{\alpha} \equiv - 3 \left\{ \kappa_{L2} h_L^{\alpha}(\nu_L) + \sum_A (M_{LA4} h_A^{\alpha} + M_{LA5} q_A^{\alpha}) \right\} \quad ; \tag{7.111}$$

$$\tilde{D}_L^{\alpha\beta} \equiv - \frac{15}{2} \left\{ \kappa_{L3} \pi_L^{\alpha\beta}(\nu_L) + \sum_A M_{LA6} \pi_A^{\alpha\beta} \right\} \quad . \tag{7.112}$$

The expression above for $D_{coll} N_L$ is all that we require to solve for the massless particle case transport equations.

In summary, we have developed in this chapter three sets of equations. The first set describes $T^{\alpha\lambda}_{A|\lambda}$ and $U^{\alpha\beta\lambda}_{A\ |\lambda}$ of the massive particle components excluding interactions with massless particles. The second set of equations describes the additional terms which we have to include in $T^{\alpha\lambda}_{A|\lambda}$ and $U^{\alpha\beta\lambda}_{A\ |\lambda}$ if we wish to include the reactions of massive particles with massless particles. Finally, the third set of equations describes $D_{coll}N_L$ for massless particles reacting with massive particles, reactions with other massless particles being considered as negligible. We have also shown that the plausible expansion of $T^{\alpha\lambda}_{A|\lambda}$, $U^{\alpha\beta\lambda}_{A\ |\lambda}$, and $D_{coll}N_L$ in terms of the deviations from equilibrium is quite valid; furthermore, the calculation indicates how the scalar coefficients are related to the transition probabilities.

VIII. Summary

In this thesis we have developed a non-stationary theory of mixtures within the context of Boltzmann kinetic theory. The two major results that we derived are, for each component of the gas, the expression for the entropy production, and the transport equations which describe the behaviour of the thermal and viscous effects. These results are important because the entropy production and the transport equations, in conjunction with the balance equations for the mass flux and the energy-momentum tensor, are just the equations we need to describe the macroscopic behaviour of the gas.

The transport equations indicate the synergistic effects in the gas. Not only do the thermal fluxes and viscous stresses of a particular species depend upon purely thermodynamical quantities defined with respect to that species, but they also depend upon the thermodynamical quantities, heat fluxes, and viscous stresses of all the other species in the gas. In addition, the transport equations include relaxation terms to describe the transient effects in the gas. These relaxation terms also appear in the entropy flux and imply that, for any initial deviation from equilibrium, the entropy is reduced, i.e. equilibrium is a local maximum. This is a manifestation of positive relaxation times.

Another feature of the transport equations is their insensitivity to the nature of the collisional terms appearing in them. The transport equations may be written schematically as TD=COLL where the lefthand side, TD (thermodynamical), contains thermodynamical functions, their derivatives, heat fluxes, and viscous stresses etc. The righthand side, COLL (collisional), are the collisional structures formed from a linear combination of the macroscopic deviations from equilibrium and coefficients determined from the collision cross-sections. While these collisional coefficients are sensitive to the nature of the collisions involved in the gas, the general algebraic stucture of COLL is not. Furthermore, our main result, for both the massive and massless cases, is the specification of TD for all the transport equations. These structures are independent of the details appearing in COLL and hence are applicable to all situations whereas COLL will change according to the system being considered.

Our theory of a multi-component gas with transient effects is a generalization of, and an improvement over, the single species non-stationary theory and the quasi-stationary multi-component theory. Under the special conditions of a gas consisting of only one massive particle species, our theory is identical to the non-stationary theory of Israel and Stewart [16]. On the other hand, when we neglect the transient terms in our theory we produce, assuming the vorticity is zero, $\omega^{\alpha\beta}= 0$, the stationary multi-component theory of Stewart [33]. Finally, for the massless case, when the deviations from equilibrium for the matter may be neglected altogether, we obtain the quasi-stationary results of Straumann [35] when we neglect the transient terms in our theory.

Our theory is also an improvement over the quasi-stationary multi-component case and the single component transient case because it has a wider range of applicability. The situations envisioned here are cases where a multicomponent approach is necessary and where a quasi-stationary theory is inappropriate. Possible applications for our theory which come to mind are of an astrophysical nature: the accretion of matter through the boundary between a star and a hypothetical neutron star or black hole imbedded in the star's core (Thorne and Zytkow [38]); accretion disks around black holes and neutron stars; and the leptonic era in the early history of the universe.

In conclusion, therefore, we have a non-stationary theory of a multi-component gas containing massive and massless particles which is a significant improvement over previous theories; and this theory is applicable to situations of continued interest to astrophysicists.

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Appendix A: The Standard Thermodynamic Functions

Let us consider the integration of I_{Anq} . From equation (3.5) we have that

$$I_{nq} = \frac{(-1)^n}{m^{n-1}(2q+1)!!} \int \Big\{ \mathring{N}\, (u_\alpha p^\alpha)^{n-2q} \times \qquad \text{(A1)}$$

$$[p^\alpha p_\alpha + (u^\alpha p_\alpha)^2]^{2q}\, \delta(p^\alpha p_\alpha + \mathring{m}^2)\theta(p^4)\Big\} d^4p^\alpha \quad .$$

Let us choose a local Lorentz frame such that $u^\alpha = (0,0,0,1)$. In this frame we have $p^\alpha = (p\cos\phi\sin\theta, p\sin\phi\sin\theta, p\cos\theta, E)$. This implies that $p^\alpha p_\alpha = E^2 - p^2$ and $u_\alpha p^\alpha = -E$. Then we have $d^4p^\alpha = E^2 \sin\theta d\theta d\phi dE dp$ where $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi$, $0 \leq p, E \leq \infty$. Then equation (A1) becomes

$$I_{nq} = \frac{(-1)^n 4\pi g}{m^{n-1}(2q+1)!!} \int \Big\{ \frac{1}{e^{\tilde{\beta}E-\alpha} - \varepsilon}\, (-E)^{n-2q} \times \qquad \text{(A2)}$$

$$p^{2q+2}\, \delta(p^2 - E^2 + \mathring{m}^2)\theta(E) dp dE \quad .$$

Integration over E gives us

$$I_{nq} = \frac{(-1)^n 4\pi g}{m^{n-1}(2q+1)!!} \int_0^\infty \frac{(p^2 + \mathring{m}^2)^{[(n-2q-1)/2]} p^{2q+1}}{\exp\{\tilde{\beta}(p^2 + \mathring{m}^2)^{1/2} - \alpha\} - \varepsilon}\, dp \qquad \text{(A3)}$$

For $\mathring{m} \neq 0$ we set $p = \mathring{m}\sinh\chi$ so that $dp = \mathring{m}\cosh\chi d\chi$. Then $(p^2 + \mathring{m}^2)^{1/2} = \mathring{m}\cosh\chi$. For $\mathring{m} = 0$ we have $(p^2 + \mathring{m}^2) = p^2$ and we set $\chi = \tilde{\beta}p$. In both cases $0 \leq \chi \leq \infty$. Equation (A3) now becomes

$$I_{nq} = \begin{cases} \dfrac{4\pi g m^3}{(2q+1)!!}\displaystyle\int_0^\infty \frac{(\sinh\chi)^{2q+2}(\cosh\chi)^{n-2q}}{\exp(\beta\cosh\chi - \alpha) - \varepsilon}\,d\chi\ , & \mathring{m} \neq 0 \\[2ex] \dfrac{4\pi g}{\beta^{n+2}(2q+1)!!}\displaystyle\int_0^\infty \frac{\chi^{n+1}}{\exp(\chi - \alpha) - \varepsilon}\,d\chi\ , & \mathring{m} = 0 \end{cases} \qquad \text{(A4)}$$

Repetition of the above calculation for J_{nq} gives us

$$J_{nq} = \begin{cases} \dfrac{4\pi g m^3}{(2q+1)!!}\displaystyle\int_0^\infty \frac{e^{\beta\cosh\chi - \alpha}(\sinh\chi)^{2q+2}(\cosh\chi)^{n-2q}}{[\exp(\beta\cosh\chi - \alpha) - \varepsilon]^2}\,d\chi\ , & \mathring{m} \neq 0 \\[2ex] \dfrac{4\pi g}{\beta^{n+2}(2q+1)!!}\displaystyle\int_0^\infty \frac{e^{\chi-\alpha}\ \chi^{n+1}}{[\exp(\chi-\alpha) - \varepsilon]^2}\,d\chi\ , & \mathring{m} = 0 \end{cases} \qquad \text{(A5)}$$

For massive particles, the integrals for I_{nq} and J_{nq} can be expressed in terms of a standard set of integrals $K_n(\alpha, \beta)$, $L_n(\alpha, \beta)$ where [12,16]

$$K_n(\alpha,\beta) \equiv \frac{\beta^n}{(2n-1)!!}\int_0^\infty \frac{(\sinh\chi)^{2n}}{\exp(\beta\cosh\chi - \alpha) - \varepsilon}\,d\chi\ ; \qquad \text{(A6)}$$

$$L_{n+1}(\alpha,\beta) \equiv \frac{\beta^n}{(2n-1)!!}\int_0^\infty \frac{(\sinh\chi)^{2n}\cosh\chi}{\exp(\beta\cosh\chi - \alpha) - \varepsilon}\,d\chi\ . \qquad \text{(A7)}$$

For n = 1, 2, ... we have

$$\left(\frac{\partial K_n}{\partial\alpha}\right)_\beta = L_n\ ; \qquad \text{(A8)}$$

$$\left(\frac{\partial L_{n+1}}{\partial \alpha}\right)_{\beta} = -\beta^{n}\frac{\partial}{\partial \beta}(\beta^{-n}K_{n}) = K_{n-1} + \frac{2n}{\beta}K_{n} \quad ; \tag{A9}$$

and for n = 2, 3, ... we have

$$-\beta^{n}\frac{\partial}{\partial \beta}(\beta^{-n}L_{n}) = L_{n-1} + \frac{2n}{\beta}L_{n} \quad . \tag{A10}$$

Consequently, for $q \le n/2$, n = 0, 2, 4, ...

$$I_{nq} = 4\pi g m^{3}\sum_{r=0}^{(n/2)-q}\frac{(2q+2r+1)!!}{(2q+1)!!}\binom{\frac{n}{2}-q}{r}K_{q+r+1}\beta^{-(q+r+1)} \quad ; \tag{A11}$$

$$I_{n+1,q} = 4\pi g m^{3}\sum_{r=0}^{(n/2)-q}\frac{(2q+2r+1)!!}{(2q+1)!!}\binom{\frac{n}{2}-q}{r}L_{q+r+1}\beta^{-(q+r+1)}; \tag{A12}$$

for n = 0, 2, ...

$$J_{nq} = 4\pi g m^{3}\sum_{r=0}^{(n/2)-q}\frac{(2q+2r+1)!!}{(2q+1)!!}\binom{\frac{n}{2}-q}{r}L_{q+r+1}\beta^{-(q+r+1)} \quad ; \tag{A13}$$

and finally for n= 1, 3, ...

$$J_{nq} = 4\pi g m^{3}\sum_{r=0}^{(n/2)-q}\frac{(2q+2r+1)!!}{(2q+1)!!}\left\{ (n+1)\binom{\frac{n-1}{2}-q}{r} + (2q+1)\binom{\frac{n-1}{2}-q}{r}\right\} K_{q+r}\,\beta^{-(q+r+1)} \quad . \tag{A14}$$

The I_{nq}'s and J_{nq}'s obey some useful recursion relations. Algebraic manipulation of equations (A4) and (A5) give us

$$I_{n+2,q} = \tilde{\varepsilon} I_{nq} + (2q+3)I_{n+2,q+1} \quad ; \tag{A15}$$

$$J_{n+2,q} = \tilde{\varepsilon} J_{nq} + (2q+3) J_{n+2,q+1} \quad ; \qquad \text{(A16)}$$

where $\tilde{\varepsilon}=0$ for $\mathring{m}=0$ and $\tilde{\varepsilon}=1$ for $\mathring{m} \neq 0$.

Integration by parts of equation (A4) produces the result, valid for all types of mass, that

$$J_{n+1,q} = \frac{(n-2q+1) I_{nq} + I_{n,q-1}}{\beta} \quad ; \qquad \text{(A17)}$$

or alternatively

$$J_{n+1,q} = \frac{(n+2) I_{nq} + \tilde{\varepsilon} I_{n-2,q-1}}{\beta} \quad . \qquad \text{(A18)}$$

Since the I_{nq}'s and J_{nq}'s are functions of α, β, we can obtain differential relations:

$$dI_{nq} = J_{nq} d\alpha - J_{n+1,q} d\beta \quad ; \qquad \text{(A19)}$$

$$dJ_{nq} = \frac{(n+1) J_{n-1,q} + J_{n-3,q-1}}{\beta} d\alpha - \frac{(n+2) J_{nq} + J_{n-2,q-1}}{\beta} d\beta ; \qquad \text{(A20)}$$

where the second equation is obtained by differentiation of equation (A17).

We define the following functions for mathematical convenience:

$$D_{nq} \equiv J_{n-1,q} J_{n+1,q} - J_{nq}^2 \quad ; \qquad \text{(A21)}$$

$$\eta \equiv (I_{20} + I_{21})/I_{10} = J_{31}/J_{21} \quad . \tag{A22}$$

From equation (A4) and (A5) we may express I_{nq} and J_{nq} for massless particles as

$$I_{nq} = \int_0^\infty I_{nq}(\nu)d\nu \quad ; \tag{A23}$$

$$J_{nq} = \int_0^\infty J_{nq}(\nu)d\nu \quad ; \tag{A24}$$

where

$$I_{nq}(\nu) = \frac{4\pi g}{\beta^{n+2}(2q+1)!!} \, \frac{\nu^{n+1}}{e^{\nu-\alpha} - \varepsilon} \quad ; \tag{A25}$$

$$J_{nq}(\nu) = \frac{4\pi g}{\beta^{n+2}(2q+1)!!} \, \frac{e^{\nu-\alpha}\,\nu^{n+1}}{\{e^{\nu-\alpha} - \varepsilon\}^2} \quad . \tag{A26}$$

Equations (A25) and (A26) are the specific forms of $I_{nq}(\nu)$ and $J_{nq}(\nu)$ as defined by equations (6.15) and (6.16).

Appendix B: List of Symbols

a_A	-a deviation from equilibrium appearing in f_A (p. 42).
a_{nq}	-a numerical coefficient (p. 34).
b_A	-a deviation from equilibrium appearing in f_A (p. 43).
b_A^{λ}	-a deviation from equilibrium appearing in f_A (p. 43).
$\tilde{b}_A^{\lambda}$	-a deviation from equilibrium appearing in f_A (p. 42).
$B(N_A)$	-the collisional terms due to binary collisions alone (p. 91).
c	-the speed of light (=1 in this thesis)(p. 17).
c_A	-a deviation from equilibrium appearing in f_A (p. 43).
c_A^{λ}	-a deviation from equilibrium appearing in f_A (p. 43).
$c_A^{\alpha\beta}$	-a deviation from equilibrium appearing in f_A (p. 43).
$\tilde{c}_A^{\alpha\beta}$	-a deviation from equilibrium appearing in f_A (p. 42).
$C^{\sim}_{V,\bar{n}}$	-the specific heat at constant volume (p. 39).
$C_{P,\bar{n}}$	-the specific heat at constant pressure (p. 39).
$C_{ABC}^{\alpha(n)}(\xi_A)$	-a general collision integral (p. 126) (p. 134).
$d\Sigma$	-the space-like element of three volume (p. 20).
dV_A	-the volume element in momentum space (p. 20).

$d\Omega_L$	-the element of solid angle (p. 93).
D_{nq}	-a specific combination of J_{nq}'s (p. 39).
$D_L^{\alpha(n)}(\nu_L)$	-a coefficient in the expansion of the Boltzmann equation for massless particles (p. 105).
$\tilde{D}_L^{\alpha(n)}(\nu_L)$	-a coefficient in the expansion of the Boltzmann equation for massless particles (p. 145).
$D_{coll}N_A$	-the collision term in the Boltzmann equation (p. 23).
e_A	-the electric charge of a particle (p. 19).
f_A	-the first order deviation from equilibrium expressed as a power series in p_A^α(p. 42).
g	-the determinant of the metric tensor (p. 20).
g_A	-the spin weight factor (p. 22).
$g_{\alpha\beta}$	-the metric tensor (p. 17).
h	-Planck's constant (equals one in this thesis) (p. 18).
h_A^α	-the momentum flux (p. 30) (p. 11).
$h_L^\alpha(\nu_L)$	-the spectral momentum flux (p. 94).
H	-the enthalpy (p. 37).
$I_A{}^{\alpha(n)}$	-a standard integral (p. 34).
I_{Anq}	-a standard thermodynamic function of α_A and β_A (p. 35).
j_A^α	-the particle drift (p. 30).
$j_A^\alpha(\nu_L)$	-the spectral particle drift (p. 94).
J_{Anq}	-a standard thermodynamic function of α_A and β_A (p. 35).
k	-Boltzmann's constant (p. 37).

$K^{\alpha(n)}_{LA}(p^{\alpha}_{L})$	-a frequency dependent collision integral (p. 140).
$K^{\alpha(n)}_{LABC}(p^{\alpha}_{L})$	-a frequency dependent collision integral (p. 140).
$K_{L}(p^{\alpha}_{L})$	-a frequency dependent collision integral (p. 140).
$\bar{K}^{\lambda}_{LL}(p^{\alpha}_{L})$	-a frequency dependent collision integral (p. 141).
$\bar{K}^{i}_{LA}$	-a frequency dependent collision coefficient (p. 141).
K'^{i}_{LA}	-a frequency averaged collision coefficient (p. 144).
k^{α}_{L}	-an arbitrary spatial vector of unit length (p. 93).
L_{ABnq}	-a collision coefficient (p. 128).
L_{ABi}	-a collision coefficient (p. 130).
$L_{AB\alpha(n)}$	-a collision integral (p. 127).
$\overset{\circ}{m}_{A}$	-the rest mass of a particle (p. 19).
m_{A}	-a generalized mass (p. 20).
M^{α}_{A}	-the mass flux (p. 25).
$\overset{\circ}{M}{}^{\alpha}_{A}$	-the zeroth order part of the mass flux (p. 46).
$M_{LAi}(\nu_{L})$	-a spectral collision coefficient (p. 143).
$\bar{M}_{LAi}$	-a frequency averaged collision coefficient (p. 109).
n_{A}	-the mass density (coincides with number density for massless particles) (p. 29).
$\bar{n}_{A}$	-the number density (p. 35).

$\bar{n}_A$ -the fractional proportion by mass of species A (p. 37).
$\overset{*}{n}$ -the number of species in the gas (p. 60).
$n_L(\nu_L)$ -the spectral number density (p. 94).
n_μ -the time-like normal to the spatial surface $d\Sigma$ (p. 22).
N_A -the distribution function (p. 20).
$\overset{\circ}{N}_A$ -the equilibrium distribution function (p. 32).
$N_L(\nu_L)$ -the spectral distribution function for massless particles (p. 97).
N^α_A -the number flux (p. 25).
$N^\alpha_L(\nu_L)$ -the spectral number flux (p. 44).
p^α_A -the four momentum of a particle (p. 20).
P_A -the thermodynamic pressure (p. 30).
$\tilde{P}_A$ -the bulk pressure (p. 30).
q^α_A -the heat flux (p. 48).
Q^α_A -the second order tensor appearing in the entropy flux (p. 56).
Q^i_A -the thermodynamic functions appearing in Q^α_A (p. 58).
s_A -the scalar spin of a particle (p. 19).
S_A -the entropy (p. 36).
S^α_A -the entropy flux (p. 27).
S^α -the total entropy flux of the gas (p. 27).
T_A -the temperature of species A (p. 33).

T_E	-the Eckart temperature (p. 99).
T_M	-the matter temperature (p. 99).
T^{i}_{AL}	-a frequency dependent collision term (p. 137).
$T^{\alpha\beta}_{A}$	-the energy-momentum tensor (p. 26).
$T^{\alpha\beta}$	-the total energy-momentum tensor of the gas (p. 26).
$\overset{\circ}{T}{}^{\alpha\beta}_{A}$	-the zeroth order part of the energy-momentum tensor (p. 47).
$T^{\alpha\beta}_{L}(\nu_L)$	-the spectral energy-momentum tensor (p. 97).
t_{Aij}	-the thermodynamic functions appearing in the transport equations (p. 83).
$T(N_L)$	-the collision term in the Boltzmann equation for massless particles due to annihilation and creation processes (p. 91).
$\overset{*}{T}(N_A)$	-the collision term in the Boltzmann equation for massive particles due to creation and annihilation processes (p. 92).
U	-the internal energy (p. 37).
u^{α}	-the unit flow vector or the four velocity of a comoving observor (p. 24).
u^{α}_{A}	-the unit flow vector for species A (p. 42).
U_{Ai}	-the thermodynamic functions associated with the double momentum flux (p. 54).
U^{i}_{AL}	-a frequency dependent collision term (p. 137).
$U^{\alpha\beta\gamma}_{A}$	-the double momentum flux (p. 27).
$\overset{\circ}{U}{}^{\alpha\beta\gamma}_{A}$	-the zeroth order part of the double momentum flux (p. 54).

$\tilde{v}$	-an arbitrary parameter; also, a specific volume (p. 37).
w^{α}_{A}	-the four velocity of a particle (p. 20).
$W(p_A,p_B\|\overset{*}{p}_A,\overset{*}{p}_B)$; W_{AB}	-transition probability for binary collisions (p. 22).
$W(p_A,p_B\|\overset{*}{p}_A,\overset{*}{p}_B,p_L)$; W_{ABL}	-transition probability for creation processes (p. 91).
$W(p_A,p_B,p_L\|\overset{*}{p}_A,\overset{*}{p}_B)$; W_{LAB}	-transition probability for annihilation processes (p. 91).
x^{α}	-the space-time coordinates (p. 18).
y_A	-the natural logarithm of N_A/Δ_A (p. 42).
$\mathring{y}_A$	-the natural logarithm of $\mathring{N}_A/\mathring{\Delta}_A$ (p. 42).
α_A	-the thermal potential (p. 32).
$\tilde{\alpha}$	-the coefficient of volume expansion (p. 38).
β_A	-the mass times the inverse temperature (p. 32).
β_E	-the mass times the inverse of the Eckart temperature (p. 99).
β_M	-the mass times the inverse of the matter temperature (p. 99).
$\tilde{\beta}_A$	-the inverse of kT_A , referred to as the inverse temperature (p. 42).
$\tilde{\beta}^{\alpha}_{A}$	-the inverse temperature times the flow vector (p. 42).
β^{α}_{A}	-the flow vector times β_A (p. 42).
γ	-the adiabatic index (p. 40).
δ^{α}_{A}	-the difference between u^{α}_{A} and a common flow vector u^{α} (p. 44).

δf_A	-a fitting and frame change of f_A (p. 61).
δn_A	-the first order part of the mass density (p. 61).
δN_A	-the first order part of the distribution function (p. 44).
δM_A^α	-the first order part of the mass flux (p. 46).
$\delta T_A^{\alpha\beta}$	-the first order part of energy-momentum tensor (p. 47).
δu_A^α, δu^α	-a frame change of u_A^α , u^α (p. 43).
$\delta U_A^{\alpha\beta\gamma}$	-the first order part of the double momentum flux (p. 54).
$\delta(x)$	-the Dirac delta function (p. 20).
$\delta\alpha_A$	-a fitting change of the thermal potential (p. 43).
$\delta\beta_A$	-a fitting change of β_A (p. 43).
$\delta\rho_A$	-the first order part of the energy density (p. 47).
Δ_A	-the Bose enhancement or Fermi exclusion function for collisions (p. 22).
$\Delta_{\alpha\beta}$	-the spatial projection operator of u^α (p. 29).
ΔN_A	-a fitting and frame change of the distribution function (p. 61).
ζ_A	-a variable defined by a contraction of the double momentum tensor (p. 53).
η_A	-the relativistic enthalpy (p. 38).
$\bar{\eta}_A$	-a thermodynamic function defined via a derivative of η_A with respect to β_A (p. 67).

$\eta_{\alpha\beta}$ -the Minkowski flat space metric (p. 17).

θ -the volume expansion; an angular variable in integrals (p. 24).

$\theta(x)$ -the heaviside step function (p. 20).

Θ_A -the thermal potential per unit mass (p. 36).

κ -the isothermal compressibility (p. 38).

κ_{Li} -a spectral coefficient of absorption (p. 145).

$\bar{\kappa}_{Li}$ -a frequency averaged coefficient of absorption (p. 108).

λ -the mean free path (p. 64).

λ_L -the coefficient of thermal conductivity for massless particles (p. 112).

λ_{AB} -the heat conductivity matrix (p. 88).

Λ_A -a thermodynamic function (p. 48).

ν_b -the bulk viscosity for massless particles (p. 116).

ν_L -the frequency (p. 93).

ν_{AB} -the shear viscosity matrix (p. 88).

$\tilde{\nu}_L$ -the shear viscosity for massless particles (p. 112).

$\tilde{\nu}_{AB}$ -the bulk viscosity matrix (p. 87).

ξ_A -a combination of Ψ_A and Φ_A(p. 23).

Ξ -a function of the distribution function (p. 28).

π - 3.14159...; the bulk stress of the whole gas (p. 11).

π_A -the bulk stress of species A (p. 35).

$\Pi^{\alpha(n)}_{(q)}$ -a projection operator (p. 34).

ρ_A	-the energy density (p. 30).
$\rho_L(\nu_L)$	-the spectral energy density (p. 94).
σ	-the entropy per unit mass, the cross-section (p. 87).
$\sigma^{\alpha\beta}$	-the shear tensor (p. 30).
τ	-a particle's world line parameter, the optical depth (p. 20).
υ_A	-a variable defined by a contraction of the double momentum flux (p. 53).
ϕ	-an angular variable in integrals (p. 153).
Φ_A	-a function solely of the distribution function (p. 23).
χ	-a hyperbolic angle variable in integrals (p. 153).
$\chi_A^{\alpha(n)}$	-an arbitrary thermodynamic function of position (p. 65).
$\bar{\chi}_A$	-an arbitrary thermodynamic function (p. 66).
χ_{ABi}	-a collision coefficient (p. 85).
$\tilde{\chi}_{ABi}$	-a collision coefficient (p. 131).
χ_{ABnq}	-a collision coefficient (p. 129).
Ψ_A	-an arbitrary tensor function of position and momentum (p. 23).
$\omega^{\alpha\beta}$	-the vorticity tensor (p. 30).
Ω_A	-a thermodynamic function (p. 50).
Ω_{Aij}	-a thermodynamic function associated with the solution for the deviations from equilibrium (p. 51).

Special Symbols

∇_μ	-the spatial derivative (p. 66).
"•"	-the time derivative (p. 66).
"_"	-over thermodynmical functions it represent a derivative with respect to the inverse temperature; over collision coefficients it represents a frequency avaerage (p. 66 , p. 109).